

NODAL LENGTH OF STEKLOV EIGENFUNCTIONS ON REAL-ANALYTIC RIEMANNIAN SURFACES

IOSIF POLTEROVICH, DAVID A. SHER, AND JOHN A. TOTH

ABSTRACT. We prove sharp upper and lower bounds for the nodal length of Steklov eigenfunctions on real-analytic Riemannian surfaces with boundary. The argument involves frequency function methods for harmonic functions in the interior of the surface as well as the construction of exponentially accurate approximations for the Steklov eigenfunctions near the boundary.

1. INTRODUCTION AND MAIN RESULTS

1.1. Steklov problem. Let Ω be a compact n -dimensional manifold with C^∞ boundary $\partial\Omega$ and unit exterior normal ν . We consider the Steklov eigenvalue problem

$$\begin{aligned}\Delta u_\lambda(x) &= 0, \quad x \in \Omega, \\ \partial_\nu u_\lambda(q) &= \lambda u_\lambda(q), \quad q \in \partial\Omega.\end{aligned}\tag{1.1.1}$$

The solutions $u_\lambda \in C^\infty(\Omega)$ are called Steklov eigenfunctions. Let $\gamma_{\partial\Omega} : C(\Omega) \rightarrow C(\partial\Omega)$ denote the boundary restriction map. The boundary restrictions of the Steklov eigenfunctions, $\varphi_\lambda := \gamma_{\partial\Omega} u_\lambda$, are the eigenfunctions of the Dirichlet-to-Neumann (sometimes also referred to as the Poincaré–Steklov) operator P . It is well-known that P is a first-order homogeneous, self-adjoint, elliptic pseudodifferential operator $P \in \Psi^1(\partial\Omega)$, where $P - \sqrt{-\Delta_{\partial\Omega}} \in \Psi^0(\partial\Omega)$ (see [Ta1]). Consequently, the Steklov spectrum, which coincides with the spectrum of P , consists of an infinite sequence of eigenvalues λ_j with $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$.

Recently, there has been a significant interest in the study of the nodal geometry for the Steklov problem, see [BL, Ze, WZ]. However, these results were concerned with the nodal sets of the Dirichlet-to-Neumann eigenfunctions φ_λ , i.e. the boundary nodal sets. The present paper focusses on the interior nodal sets, i.e. the nodal sets of the Steklov eigenfunctions u_λ . With the exception of the recent preprint [SWZ] (see Remark 1.2.3), very little has been previously known on this subject. We also note that Courant type bounds were earlier obtained on the number of the interior (in any dimension) and boundary (in dimension two) nodal domains of Steklov eigenfunctions ([KS, AM, KKP], see also [GP, section 6.1]).

1.2. Main result. Motivated by a celebrated conjecture of Yau [Y1, Y2], it has been recently suggested in [GP, Open problem 11 (i)] that the ratio between the $(n-1)$ -dimensional Hausdorff measure of the nodal set of a Steklov eigenfunction u_λ and the corresponding Steklov eigenvalue λ is bounded above and below by some positive constants depending only on the geometry of the manifold. Our main result below gives a positive answer to this conjecture for real-analytic Riemannian surfaces. Note that, in this case, the 1-dimensional Hausdorff measure of a nodal set Z_{u_λ} is simply equal to its total length, which we will denote by $\mathcal{L}(Z_{u_\lambda})$.

THEOREM 1.2.1. *Let Ω be a real-analytic compact Riemannian surface with boundary. Then there exist constants $C_j = C_j(\Omega) > 0$, $j = 1, 2$, such that, for any Steklov eigenfunction u_λ with an eigenvalue $\lambda > 0$, the total length of its nodal set Z_{u_λ} satisfies:*

$$C_1\lambda \leq \mathcal{L}(Z_{u_\lambda}) \leq C_2\lambda.$$

This result may be viewed as an analogue of the Donnelly-Fefferman bound for the size of the nodal set of Laplace eigenfunctions on real-analytic manifolds [DF].

Remark 1.2.2. To illustrate the nodal bounds in Theorem 1.2.1, consider the Steklov eigenfunctions on a unit disk corresponding to the double eigenvalue $\lambda_{2n-1} = \lambda_{2n} = n$, $n = 1, 2, \dots$. They can be represented in polar coordinates by $u(r, \theta) = r^n \sin(n\theta + \alpha)$ for some $\alpha \in [0, \pi/2]$. The nodal set of each such eigenfunction is a union of n diameters, of total length $2n$.

Remark 1.2.3. Our methods here are specific to the case of real-analytic surfaces. In higher dimensions, Steklov eigenfunctions on Ω are much more complicated and their nodal structures are not well-understood. In the case of a general smooth n -dimensional manifold with smooth boundary, a non-sharp lower bound

$$|Z_\lambda| \geq c\lambda^{\frac{2-n}{2}}$$

for the volume $|Z_\lambda|$ of the interior nodal sets has recently been proven in [SWZ]. It can be viewed as the Steklov analogue of the lower bounds on the size of the nodal sets of Laplace eigenfunctions obtained in [CM, SZ, HS, M, HW].

After the first version of the present paper was posted on the archive, an upper bound of order $\lambda^{3/2}$ on the size of the nodal set of Steklov eigenfunctions on surfaces was obtained in [Zh2] using a quite different approach. While this bound is not sharp, it is valid for arbitrary compact smooth surfaces with boundary. More recently, while the present paper was under review, the sharp upper bound in Theorem 1.2.1 has been extended in [Zh3] to real-analytic Riemannian manifolds of arbitrary dimension.

1.3. Sketch of the proof. For any closed manifold with boundary Ω , Steklov eigenfunctions u_λ decay rapidly into the interior of Ω [HL, GPPS]. In order to analyze their nodal lengths, we must therefore consider a neighborhood of the boundary and its complement separately. The idea is to use quasimodes near the boundary and frequency function techniques in the interior.

We begin, in section 2, by constructing quasimodes for the Dirichlet-to-Neumann operator P . In Proposition 2.3.4, we show that trigonometric functions are approximate eigenfunctions for P , up to an exponentially decaying error term. The proof uses the C^ω surface assumption in a very strong way: from [GPPS], we conclude that the error term decays faster than any polynomial in λ , and the assumption of analyticity allows us to improve the decay of the error term to exponential. From Proposition 2.3.4 and some fairly standard linear algebra techniques, we show in Lemma 3.1.1 that the boundary Steklov eigenfunctions φ_λ can be approximated by trigonometric functions ψ_λ plus an error f_λ which decays exponentially in λ in any C^k norm.

We would like to use the quasimode approximation to estimate nodal length of a Steklov eigenfunction u_λ near the boundary, but from the example of the annulus (see Example 1.4), we know that it is possible for u_λ to itself be exponentially small near one or more

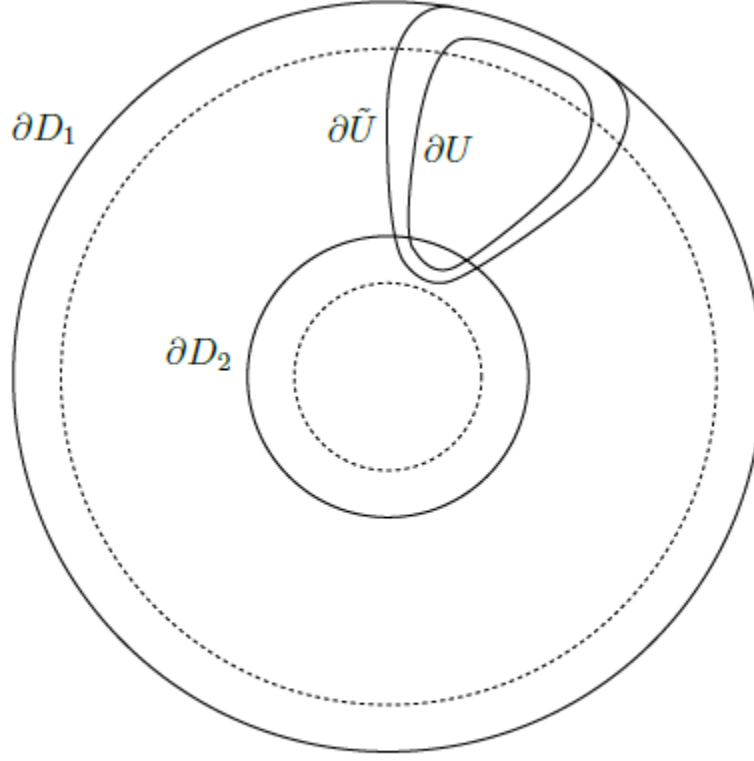


FIGURE 1. Illustration of the proof. The domain D is the annulus between the boundary components ∂D_1 and ∂D_2 , denoted by solid lines. The dotted circles mark a neighborhood of ∂D_1 in D and a neighborhood of ∂D_2 in D^c . The two sets $U \subset \tilde{U}$ are as illustrated.

boundary components. Indeed, this seems to be the generic situation. Near these “residual” boundary components, the error in the quasimode approximation can be larger than the quasimode itself and hence the approximation is not particularly useful. Our proof avoids this difficulty via harmonic extension of the Steklov eigenfunctions across residual boundary components. By controlling the C^k -norm of the extended eigenfunctions (see Lemma 4.3.1), one can effectively treat a neighborhood of the “residual” boundary components in the same way as the interior of D .

To illustrate the proof, consider a Steklov eigenfunction u_λ on the annular domain D in Figure 1. Suppose that ∂D_1 is a non-residual, i.e. “dominant”, boundary component for u_λ and that ∂D_2 is a residual boundary component. In section 3.2, we extend our boundary quasimode approximation given by Lemma 3.1.1 into the interior. The exponential decay of the error means that the approximation is effective in a λ -independent neighborhood of ∂D_1 . A direct comparison of nodal sets (section 4.2) then gives us upper and lower bounds on the nodal length of u_λ in a small neighborhood of ∂D_1 , denoted by the dotted line in Figure 1.

To treat the interior and a neighborhood of the residual boundary components, we extend u_λ (section 2.4) to a small λ -independent neighborhood of ∂D_2 , with boundary denoted by the dotted circle. In this neighborhood, u_λ is still exponentially small in λ (section 4.3). Now let $U \subset \tilde{U}$ be two sets as in Figure 1. In section 4.4, we use standard frequency function techniques to bound the nodal length of u_λ in U from above by the “renormalized Almgren frequency function” of the larger domain \tilde{U} . Then, using the quasimode approximation near the boundary and the fact that a portion of $\partial\tilde{U}$ coincides with ∂D_1 , we bound this frequency function by a multiple of λ . A covering argument extends these upper bounds to all of D outside a neighborhood of ∂D_1 , including a neighborhood of ∂D_2 . Combining these upper bounds with the two-sided bounds near ∂D_1 completes the proof.

Remark 1.3.1. A particularly novel aspect of the problem is the exponential decay into the interior of the eigenfunctions, namely that for some positive constants τ and C depending only on the geometry of Ω ,

$$|u_\lambda(x)| \leq C e^{-\tau d(x, \partial\Omega)\lambda}. \quad (1.3.2)$$

This is an immediate consequence of Lemma 3.2.9 and the exponential decay of the interior quasimode approximations. A similar estimate holds for the derivatives of order $k \geq 1$ of u_λ , provided we multiply the right-hand side by λ^k . Our problem therefore resembles the question of estimating the size of nodal sets in forbidden regions for eigenfunctions of Schrödinger operators (see [HZZ, CT]). A different method of proving (1.3.2) has been communicated to us by M. Taylor [Ta2].

Remark 1.3.3. It also follows from our results (see Proposition 3.1.3) that for real-analytic surfaces, the error term in the eigenvalue approximation of [GPPS, Theorem 1.4] in fact decays exponentially as the index of the eigenvalue increases. In particular, for a simply connected real-analytic surface Ω , there exists a constant $\tau > 0$ depending on the geometry of Ω such that

$$\lambda_{2j} = \lambda_{2j+1} + O(e^{-\tau j}) = \frac{2\pi j}{\text{Length}(\partial\Omega)} + O(e^{-\tau j}).$$

This improves the error estimate $O(j^{-\infty})$ obtained independently by Rozenblyum and Guillemin-Melrose (see [Ro, Ed]) for smooth simply connected surfaces.

1.4. Example: Steklov problem on an annulus. To illustrate several features of Steklov eigenvalues and eigenfunctions used in the proof of Theorem 1.2.1 outlined in section 1.3, consider an annulus $A(1, \varepsilon)$ with inner radius ε and outer radius 1.

First we compare its Steklov spectrum to the spectrum of a union of two disks, of radii 1 and ε , which consists of the double eigenvalue zero, as well as the double eigenvalues k and k/ε for each $k \in \mathbb{N}$. Let us show that the difference between these eigenvalues and the corresponding eigenvalues of $A(1, \varepsilon)$ is in fact exponentially vanishing in k . Indeed, as computed in [D] (see also [GP, Section 4.2]), the nonzero Steklov eigenvalues of the annulus are given by the roots of the quadratic polynomial

$$p_k(\sigma) = \sigma^2 - \sigma k \left(\frac{\varepsilon + 1}{\varepsilon} \right) \left(\frac{1 + \varepsilon^{2k}}{1 - \varepsilon^{2k}} \right) + \frac{1}{\varepsilon} k^2,$$

for each $k \in \mathbb{N}$ (each root corresponds to a double eigenvalue). We could compute the eigenvalues directly, but it is easier to compare $p_k(\sigma)$ to the polynomial

$$\tilde{p}_k(\sigma) = \sigma^2 - \sigma k \left(\frac{\varepsilon + 1}{\varepsilon} \right) + \frac{1}{\varepsilon} k^2,$$

which has roots k and k/ε . In fact $\tilde{p}_k(\sigma) - f(k, \varepsilon)\sigma = p_k(\sigma)$, where

$$f(k, \varepsilon) = k \left(\frac{\varepsilon + 1}{\varepsilon} \right) \frac{2\varepsilon^{2k}}{1 - \varepsilon^{2k}}.$$

Since $f(k, \varepsilon) = O(k\varepsilon^{2k})$, a straightforward calculation shows that the roots of $p_k(\sigma)$ differ from k and k/ε by $O(k\varepsilon^{2k})$ as well. The error is exponentially decreasing in k , which agrees with the eigenvalue approximation result in Proposition 3.1.3.

Steklov eigenfunctions on surfaces also exhibit exponential decay at certain boundary components, which motivates our definition of “dominant” and “residual” boundary components (see Definition 4.1.1). To illustrate this, consider an eigenfunction u_{σ_j} of the annulus $A(1, \varepsilon)$ corresponding to an eigenvalue $\sigma_j = k + O(k\varepsilon^{2k})$. As computed in [GP],

$$u_{\sigma_j}(r, \theta) = C_k \left(r^k + \frac{k - \sigma_j}{k + \sigma_j} r^{-k} \right) T(k\theta),$$

where $T(k\theta)$ is a unit norm linear combination of $\cos(k\theta)$ and $\sin(k\theta)$ and C_k is an appropriate normalizing constant. Since $k - \sigma_j = O(k\varepsilon^{2k})$, we see that $C_k \sim 1$, that the L^2 -norm of u_{σ_j} on $\{r = 1\}$ is roughly 1, and that the L^2 -norm on $\{r = \varepsilon\}$ is $O(\varepsilon^k)$ - which is exponentially decaying in k . So, in this case, the circle of radius 1 is a dominant boundary component, while the circle of radius ε is a residual boundary component. A similar analysis holds for the eigenfunctions corresponding to eigenvalues of the form $k/\varepsilon + O(k\varepsilon^{2k})$, with the roles of the boundary components reversed.

Acknowledgments. The authors are grateful to M. Sodin and M. Taylor for useful discussions, and to the anonymous referee for many helpful suggestions on the presentation of the paper. The research of I.P. was partially supported by NSERC, FRQNT and Canada Research Chairs Program. The research of D.S was partially supported by NSF EMSW21-RTG 1045119. The research of J.T. was partially supported by NSERC and FRQNT. I.P. and J.T. were also supported by the French National Research Agency project Gerasic-ANR-13-BS01-0007-0.

2. STEKLOV PROBLEM FOR REAL-ANALYTIC PLANAR DOMAINS

2.1. Koebe uniformization. We first prove Theorem 1.2.1 in the case where Ω is a real-analytic planar domain and explain the fairly minor modifications needed to treat the general case of compact Riemann surfaces with boundary in section 4.5. The two-dimensional case is quite special. Indeed, unlike the higher-dimensional case, for surfaces the Steklov operator $P : C^\infty(\partial\Omega) \rightarrow C^\infty(\partial\Omega)$ agrees with the square root of the Laplacian on the various boundary components modulo a smoothing operator. Upon reparametrization by arclength, the latter operator is consequently (modulo smoothing) just the Fourier multiplier M acting on the component boundary circles. Although not necessary, one can see the former directly by applying conformal mapping. Indeed, by the Koebe uniformization theorem [HS], one can conformally map Ω to a planar domain D , where D is the disk of radius 1 with a finite

number of interior disks removed. In the cases where Ω is simply-connected, this reduces to the Riemann mapping theorem. When $\partial\Omega$ is real-analytic, it follows by C^ω regularity up to the boundary in the associated Dirichlet problem for the Green's function [MN] that the conformal map $f : D \rightarrow \Omega$ extends to a real-analytic map of \overline{D} to $\overline{\Omega}$. Moreover, f maps boundary to boundary in a univalent fashion. In particular, the induced boundary restriction $f_{\partial\Omega} : \partial D \rightarrow \partial\Omega$ is a C^ω -diffeomorphism.

The boundary ∂D consists of a union of circles, which we denote by $\partial D_1, \partial D_2, \dots, \partial D_k$, with radii ρ_1, \dots, ρ_k and centers c_1, \dots, c_k respectively. The corresponding boundary components of Ω are denoted by $\partial\Omega_j := f_{\partial\Omega}(\partial D_j)$; $j = 1, 2, \dots, k$. We let $\theta_j \in [0, 2\pi]$ be the usual angle coordinate on ∂D_j for each j , and let the arc length coordinate q_j be $\rho_j \theta_j$. Let q be an arc length coordinate on ∂D which coincides with q_j when we restrict attention to ∂D_j . Finally, let dq and dq_j be the measures on ∂D and ∂D_j respectively induced by the Euclidean measure dx on \mathbb{R}^2 . As in [Ed], the Steklov problem in (1.1.1) is conformally mapped to the problem

$$\begin{aligned} \Delta v_\lambda(x) &= 0, \quad x \in D, \\ \partial_\nu v_\lambda(q) &= \lambda g v_\lambda(q), \quad q \in \partial D, \end{aligned} \tag{2.1.1}$$

with $g = df(q) = |f'(q)|$ analytic for $q \in \partial D$ and $g \neq 0$. By conformal mapping, $v_\lambda = f^* u_\lambda$ - so without loss of generality, it suffices to work with the conformal model (2.1.1) and we will do so here. We will abuse notation somewhat and denote the corresponding Steklov operator in the conformal model (2.1.1) by $P : C^\infty(\partial D) \rightarrow C^\infty(\partial D)$ as well, and its eigenfunctions by φ_λ . We henceforth make the normalization that

$$\|\varphi_\lambda\|_{L^2(\partial D)} = 1. \tag{2.1.2}$$

Since we consistently work with the model case in (2.1.1), this should not lead to confusion.

Given $\tau = (\tau_1, \dots, \tau_k) \in (0, 1)^k$, we complexify ∂D to a Grauert tube (ie. an annulus)

$$\partial D_\tau^{\mathbb{C}} = \{\oplus_{j=1}^k e^{i\theta_j - \rho_j}; (\rho_j, \theta_j) \in [0, \tau_j] \times [0, 2\pi]\}.$$

We choose $\tau_j > 0$ here to be the *analytic modulus* of $g|_{\partial D_j} \in C^\omega(\partial D_j)$; that is, the maximal tube radius for which g has a holomorphic extension to $(\partial D_j)_{\tau_j}^{\mathbb{C}}$.

We also note that the length L_j of $\partial\Omega_j = f_{\partial\Omega}(\partial D_j)$ is $\int_{\partial D_j} g(q_j) dq_j$, and define a new coordinate on ∂D_j by $s_j(q_j) = \int_0^{q_j} g(q'_j) dq'_j \in [0, L_j]$. Let s be the corresponding coordinate on all of ∂D . Note that since g is analytic and strictly positive, this is an analytic reparametrization with analytic inverse.

2.2. Potential layer formulas and the Steklov operator. We briefly review the characterization of the Steklov operator $P : C^\infty(\partial D) \rightarrow C^\infty(\partial D)$ in terms of potential layer operators. This material here is well-known and further details can be found in [Ta1, Sec. 7.1]. Here, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^∞ -boundary. Let $G(x, x') \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ be the ambient free Green's function for $-\Delta = -\sum_{j=1}^n \partial_{x_j}^2$ in \mathbb{R}^n . Consider the single and double layer operators $\mathcal{S}l : C^\infty(\partial\Omega) \rightarrow C^\infty(\Omega \cup (\mathbb{R}^n \setminus \overline{\Omega}))$ and

$\mathcal{D}l : C^\infty(\partial\Omega) \rightarrow C^\infty(\Omega \cup (\mathbb{R}^n \setminus \bar{\Omega}))$ given by

$$\begin{aligned} \mathcal{S}l f(x) &= - \int_{\partial\Omega} G(x, q) f(q) dS(q), \\ \mathcal{D}l f(x) &= - \int_{\partial\Omega} \partial_{\nu(q)} G(x, q) f(q) dS(q), \quad x \in \Omega \cup (\mathbb{R}^n \setminus \bar{\Omega}). \end{aligned} \quad (2.2.1)$$

Corresponding to $\mathcal{S}l$ and $\mathcal{D}l$ are the boundary layer operators S and $N : C^\infty(\partial\Omega) \rightarrow C^\infty(\partial\Omega)$ given by

$$\begin{aligned} S f(q) &= - \int_{\partial\Omega} G(q, q') f(q') dS(q'), \\ N f(q) &= -2 \int_{\partial\Omega} \partial_{\nu(q')} G(q, q') f(q') dS(q'), \quad q \in \partial\Omega. \end{aligned} \quad (2.2.2)$$

These operators are classical pseudodifferential with $S, N \in \Psi^{-1}(\partial\Omega)$, and S is elliptic. Given a function $v \in C^0(\mathbb{R}^n \setminus \partial\Omega)$ and $q \in \partial\Omega$, let $v_+(q)$ and $v_-(q)$ denote the limits of $v(x)$ as $x \rightarrow q$ from $x \in \Omega$ and $x \in \mathbb{R}^n \setminus \bar{\Omega}$ respectively. The layer potential operators in (2.2.1) and the induced boundary operators in (2.2.2) are linked via the boundary jumps equations

$$\begin{aligned} \mathcal{S}l f_+(q) &= \mathcal{S}l f_-(q) = S f(q), \\ \mathcal{D}l f_\pm(q) &= \pm \frac{1}{2} f(q) + \frac{1}{2} N f(q), \quad q \in \partial\Omega. \end{aligned} \quad (2.2.3)$$

Now let P be the Dirichlet-to-Neumann operator for Ω . Consider the Dirichlet problem

$$\Delta u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f.$$

A straightforward application of Green's formula gives

$$u(x) = \mathcal{D}l f(x) - \mathcal{S}l P f(x), \quad x \in \Omega. \quad (2.2.4)$$

Taking limits in (2.2.4) from within Ω and applying (2.2.3) gives

$$SP = -\frac{1}{2}(I - N). \quad (2.2.5)$$

Since $S \in \Psi^{-1}(\partial\Omega)$ and is elliptic, it follows from (2.2.5) by a parametrix construction in the standard pseudodifferential calculus that $P \in \Psi^1(\partial\Omega)$ and is also elliptic.

In the case where $\partial\Omega$ is real-analytic, P is also analytic pseudodifferential in the sense of [BK] (see also [Tr, Ch. 5]), and we write $P \in \Psi_a^1(\partial\Omega)$. To see this, we first note integration against $G(x, x')$ in \mathbb{R}^n is a pseudodifferential operator \mathcal{I} with full symbol $|\xi|^{-2}$ in the usual coordinates. As the subprincipal symbols are all zero, the symbol satisfies Cauchy estimates and so, $\mathcal{I} \in \Psi_a^{-2}(\mathbb{R}^n)$. Since $\partial\Omega$ is real-analytic, the Fermi coordinates (\bar{x}, x_n) near $\partial\Omega$ are also real-analytic. In terms of these local coordinates, $\sigma_{\mathcal{I}}(\bar{x}, x_n, \bar{\xi}, \xi_n)$ is thus an analytic symbol; hence so is $\partial_{x_n} \sigma_{\mathcal{I}}(\bar{x}, x_n, \bar{\xi}, \xi_n)$. It is easy to check that the symbols of S and N are given, respectively, by $\int_{\mathbb{R}} (\sigma_{\mathcal{I}}|_{x_n=0}) d\xi_n$ and $2 \int_{\mathbb{R}} ((\partial_{x_n} \sigma_{\mathcal{I}} + i\xi_n \sigma_{\mathcal{I}})|_{x_n=0}) d\xi_n$ respectively, and hence are also analytic. So $S, N \in \Psi_a^{-1}(\partial\Omega)$. By constructing a parametrix for S in the analytic pseudodifferential calculus (see [Tr, Ch. 5] for details), then multiplying (2.2.5) by this parametrix on the left and rearranging, it follows that

$$P \in \Psi_a^1(\partial\Omega).$$

In the following, we say that an operator $R \in \Psi_a^*(\partial\Omega)$ is *analytic smoothing* if its Schwartz kernel $K(\cdot, \cdot) \in C^\omega(\partial\Omega \times \partial\Omega)$, and we write $R \in \Psi_a^{-\infty}(\partial\Omega)$.

Remark 2.2.6. In what follows, we use the following notation: given a set X and two non-negative functions $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$, throughout the paper we write $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in X$. In addition, $f \approx g$ will mean that both $f \lesssim g$ and $g \lesssim f$ are satisfied.

2.3. Quasimodes for the Steklov operator. It is well-known ([GPPS], see also [Ed, HL]) that for smooth bounded planar domains,

$$P = M + R, \quad R \in \Psi^{-\infty}(\partial D), \quad (2.3.1)$$

where $M = \bigoplus_{j=1}^k M_j$, with $M_j : C^\infty(\partial D_j) \rightarrow C^\infty(\partial D_j)$ the Fourier multiplier defined by

$$M_j e^{2\pi i m s_j / L_j} = \frac{2\pi |m|}{L_j} e^{2\pi i m s_j / L_j}, \quad m \in \mathbb{Z}.$$

Note that $M = \sqrt{-\Delta_{\partial D}}$. As in [GPPS], let $\mathcal{A}(\alpha)$ be the sequence $\{0, \alpha, \alpha, 2\alpha, 2\alpha, \dots\}$ and $\mathcal{A}(\alpha_1, \dots, \alpha_k)$ be the union of these sequences, with multiplicity, arranged in the appropriate order. We write

$$\mu_n := \left\{ \mathcal{A} \left(\frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_k} \right) \right\}_n, \quad (2.3.2)$$

which is the n th eigenvalue of the operator M . As a consequence of (2.3.1) and [GPPS], the eigenvalues of P consist of the sequence

$$\lambda_n = \mu_n + O(n^{-\infty}), \quad n \in \mathbb{Z}^+.$$

It follows that $\lambda_n \approx n$ as $n \rightarrow \infty$. We abuse notation somewhat and denote the corresponding orthonormal basis of eigenfunctions of P on $\partial\Omega$ by $\{\varphi_{\lambda_n}\}$, so that each φ_{λ_n} is the restriction of u_{λ_n} to $\partial\Omega$. Although the operator M acting on ∂D is non-local, it clearly maps each of the boundary components ∂D_j to itself; that is, for any $f \in C^\infty(\partial D)$ with $\text{supp } f \subset \partial D_j$,

$$Mf = M_j f.$$

Denote the L^2 -normalized eigenfunctions of M on the boundary component ∂D_j by $e_{m,j}(s_j)$, where

$$e_{m,j}(s_j) = e^{2\pi i m s_j / L_j}, \quad s_j \in [0, L_j]. \quad (2.3.3)$$

We also let $e_{m,j}(s)$ be the function which is $e_{m,j}(s_j)$ on ∂D_j and 0 on the other boundary components, with an analogous definition for $\varphi_{\lambda_n,j}$.

The following lemma is central to the proof of Theorem 1.2.1 and shows that when Ω is analytic, the functions $e_{m,j}$ are quasimodes for P to exponential error in m .

PROPOSITION 2.3.4. *Suppose that Ω is analytic, with D , the conformal map $f : D \rightarrow \Omega$, and the Dirichlet-to-Neumann operator $P : C^\infty(\partial D) \rightarrow C^\infty(\partial D)$ as in (2.1.1). Then:*

- (1) *The remainder $R : C^\infty(\partial D) \rightarrow C^\infty(\partial D)$ in (2.3.1) has Schwartz kernel $R \in C^\omega(\partial D \times \partial D)$ with analytic modulus at least $\tilde{\tau} > 0$ in each variable separately. Here, $\tilde{\tau}$ depends on the analytic modulus of the conformal multiplier $g \in C^\omega$ and the geometry of D .*

(2) As $|m| \rightarrow \infty$,

$$\|Pe_{m,j}(s) - \frac{2\pi|m|}{L_j}e_{m,j}(s)\|_{C^k(\partial D)} = O_k(e^{-2\pi\tilde{\tau}|m|/L_j}).$$

In the last line, $O_k(e^{-2\pi\tilde{\tau}|m|/L_j})$ denotes a non-negative term that is $\leq C_k e^{-2\pi\tilde{\tau}|m|/L_j}$ with constant $C_k > 0$ depending on k .

Proof. Our proof uses the boundary integral equation for φ_λ . For $x, x' \in \mathbb{R}^2$, we continue to let $G(x, x')$ be the free Green's kernel for $-\Delta_{\mathbb{R}^2} = -\partial_x^2 - \partial_{x'}^2$, which is $-\frac{1}{2\pi} \log|x - x'|$. Using (2.2.5) (see also [Sh, Ro]), the functions φ_λ satisfy the boundary jumps equation:

$$\frac{1}{2}\varphi_{\lambda_n}(q) = \lambda_n \int_{\partial D} G(q, q') \varphi_{\lambda_n}(q') g(q') dq' - \int_{\partial D} K(q, q') \varphi_\lambda(q') dq', \quad (2.3.5)$$

with

$$K(q, q') = \partial_{\nu(x')} G(x, x')|_{(x, x')=(q, q')} = -\frac{1}{4\pi} \frac{\partial_{\nu(x')} |x - x'|^2}{|x - x'|^2}|_{(x, x')=(q, q')}.$$

Change variables in the first integral to integrate in terms of s' :

$$\frac{1}{2}\varphi_{\lambda_n}(q(s)) = \lambda_n \int_{\partial D} G(q(s), q(s')) \varphi_{\lambda_n}(q(s')) ds' - \int_{\partial D} K(q(s), q(s')) \varphi_\lambda(q(s')) dq(s'). \quad (2.3.6)$$

We now claim:

CLAIM 2.3.7.

$$\frac{1}{2}\varphi_{\lambda_n} = \lambda_n \tilde{G} \varphi_{\lambda_n} - \lambda_n K_1 \varphi_{\lambda_n} - K_2 \varphi_{\lambda_n},$$

where $K_1, K_2 \in \Psi_a^{-\infty}(\partial D)$ and $\tilde{G} = \oplus_{j=1}^k G_j$ with $G_j : C^\infty(\partial D_j) \rightarrow C^\infty(\partial D_j)$ given by

$$(G_j h)(s) = -\frac{1}{2\pi} \int_0^{L_j} \log |e^{2\pi i s/L_j} - e^{2\pi i s'/L_j}| h(s') ds'.$$

Recall that $\Psi_a^{-\infty}(\partial D)$ is the space of operators with C^ω Schwartz kernels. We also claim:

CLAIM 2.3.8. If we extend M_j^{-1} to act on $C^\infty(\partial D_j)$ by letting it be zero when acting on constants,

$$M_j^{-1} = 2G_j.$$

Assuming both claims, we now prove Proposition 2.3.4. The claims combine to show

$$\varphi_{\lambda_n} = \lambda_n M^{-1} \varphi_{\lambda_n} - 2\lambda_n K_1 \varphi_{\lambda_n} - 2K_2 \varphi_{\lambda_n} = M^{-1} P \varphi_{\lambda_n} - 2\lambda_n K_1 \varphi_{\lambda_n} - 2K_2 \varphi_{\lambda_n}.$$

Given $f \in C^\infty(\partial D)$, we can write $f = \sum_{k=1}^\infty \alpha_k \varphi_{\lambda_k}$ with $\alpha_k = O(k^{-\infty})$. By linearity, the boundedness of K_1 and K_2 , and the rapid decay of α_k and of $\alpha_k \lambda_k$, it follows that

$$M^{-1} P f - f = 2K_1 P f + 2K_2 f. \quad (2.3.9)$$

Since $K_1, K_2 \in \Psi_a^{-\infty}(\partial D)$ and $P \in \Psi_a^1(\partial D)$, standard theory of analytic pseudodifferential operators [Tr, Ch. 5] implies that $\tilde{K} = 2K_1 P + 2K_2 \in \Psi_a^{-\infty}(\partial D)$, with

$$M^{-1} P f - f = \tilde{K} f, \text{ or equivalently } (P - M) f = M \tilde{K} f. \quad (2.3.10)$$

Now, we choose $f(s) = e_{m,j}(s)$. For any $\tilde{\tau} < \tau_j$, by contour deformation we have

$$\begin{aligned} \tilde{K}e_{m,j}(s) &= \int_{\partial D_j} \tilde{K}(s, s_j) e^{2\pi i m s_j / L_j} ds_j \\ &= e^{-2\pi |m| \tilde{\tau} / L_j} \left(\int_{\partial D_j} \tilde{K}(s, s_j \pm i \tilde{\tau}) e^{2\pi i m s_j / L_j} ds_j \right) = O(e^{-2\pi |m| \tilde{\tau} / L_j}), \end{aligned} \quad (2.3.11)$$

where the choice of \pm is determined by the sign of m . The same is true for each $(\partial_t^k \tilde{K}e_m)(s)$, so for all k ,

$$\|\tilde{K}e_{m,j}\|_{C^k} = O_k(e^{-2\pi |m| \tilde{\tau} / L_j}).$$

The same is therefore true for $\|\tilde{K}e_{m,j}\|_{H^k}$ for any k . Since $M \in \Psi^1(\partial D)$ is a (classical) pseudodifferential operator, it is bounded from $H^k \rightarrow H^{k-1}$ for each k . From this and Sobolev embedding we conclude that $\|M\tilde{K}e_{m,j}\|_{C^k} = O_k(e^{-2\pi |m| \tau_0 / L_j})$ for any k . Combining this with (2.3.10) completes the proof of the proposition. \square

We now prove both claims.

Proof of Claim 2.3.7. Assume that $q(s) \in \partial D_j$. The portions of the integrals in (2.3.6) over $\partial D \setminus \partial D_j$ can be absorbed into K_1 and K_2 , since $d(\partial D_i, \partial D_j) > 0$ for $i \neq j$ and therefore the integrands are real-analytic functions of s' . Using the symbol \cong to denote equivalence up to terms of the form $(\lambda_n K_1 + K_2)\varphi_{\lambda_n}$ with K_1 and $K_2 \in \Psi_a^{-\infty}(\partial D)$, we have

$$\begin{aligned} \frac{1}{2}\varphi_{\lambda_n}(q_j(s_j)) &\cong \lambda_n \int_0^{L_j} G(q_j(s_j), q_j(s'_j)) \varphi_{\lambda_n}(q_j(s'_j)) ds'_j \\ &\quad - \int_{\partial D_j} K(q_j(s_j), q_j(s'_j)) \varphi_{\lambda}(q_j(s'_j)) dq_j(s'_j). \end{aligned} \quad (2.3.12)$$

For notational simplicity, we now suppress all j subscripts. The last term in (2.3.12) can be absorbed, because the normal derivative $K(q(s), q(s'))$ is itself real-analytic in $(s, s') \in [0, L] \times [0, L]$. To see this, observe that analyticity away from the diagonal $s = s'$ is clear. To establish the analyticity near $s = s'$ we insert the Taylor expansion $q(s) = q(s') + (s - s')d_{s'}q(s') + \frac{\kappa(s')}{2}|q(s) - q(s')|^2\nu(s') + [a_T(s, s')d_{s'}q(s') + a_N(s, s')\nu(s')]$ into the formula for $K(q(s), q(s'))$ in (2.3.5). Here $\kappa(s)$ denotes the curvature at the point s . The result is that

$$K(q(s), q(s')) = -\frac{1}{4\pi} \left(\frac{\kappa(s')}{2} + \frac{a_N(s, s')}{|q(s) - q(s')|^2} \right),$$

and so, $(s, s') \mapsto K(q(s), q(s')) \in C^\omega([0, L] \times [0, L])$ since a_N is real-analytic with $a_N(s, s') = O(|s - s'|^3)$ as $s \rightarrow s'$.

Finally, all we need to do to complete the proof is to show that

$$\frac{1}{2\pi} \log |q(s) - q(s')| - \frac{1}{2\pi} \log |e^{2\pi i s / L} - e^{2\pi i s' / L}| = \frac{1}{2\pi} \log \frac{|q(s) - q(s')|}{|e^{2\pi i s / L} - e^{2\pi i s' / L}|}$$

is analytic. However, the reparametrization $q \rightarrow s(q)$ is analytic with analytic inverse, and Taylor expansion of the quotient on the right-hand side shows that it is analytic and nonzero at $s = s'$ (cf. ([TZ, formula (8.8)])). Therefore, its logarithm is also analytic there. Analyticity away from $s = s'$ is automatic. \square

Proof of Claim 2.3.8. By scaling, it suffices to prove the claim for $j = 1$ and $L_1 = 2\pi$. We will show

$$-\frac{1}{\pi} \int_0^{2\pi} \log |e^{is} - e^{is'}| e^{ims'} ds' = \frac{1}{|m|} e^{ims}, \quad m \in \mathbb{Z} \setminus \{0\}. \quad (2.3.13)$$

The claim follows since $\{e^{ims}\}_{m \in \mathbb{Z}}$ is a Fourier basis. To prove (2.3.13), consider the harmonic function on the disk whose Neumann data is e^{ims} . On the one hand, this is $\frac{r^{|m|}}{|m|} e^{ims}$. On the other hand, the Poisson kernel for the Neumann problem is $-\frac{1}{\pi} \log |x - x'|$, so

$$\frac{r^{|m|}}{|m|} e^{ims} = \int_0^{2\pi} -\frac{1}{\pi} \log |re^{is} - e^{is'}| e^{ims'} ds'.$$

Restricting to $r = 1$ shows (2.3.13). \square

2.4. Harmonic extension of interior eigenfunctions across the boundary. It is well known that a harmonic function on a domain can be extended harmonically across a real analytic boundary (see [LM]). The following lemma, which will be used in the proof of Theorem 1.2.1, provides an explicit estimate on the C^k -norms of the harmonic extension of a Steklov eigenfunction u_λ across a boundary component D_j in terms of the L^2 -norm of its trace φ_λ on ∂D_j .

PROPOSITION 2.4.1. *Let $u_\lambda \in C^\omega(D)$ be a Steklov eigenfunction. Then, for every connected component ∂D_j of the boundary, and $0 < \varepsilon < \tau$ with τ defined in (3.2.1), there is a harmonic continuation of u_λ (also denoted u_λ) across the boundary component ∂D_j to an annulus $A_j^+(\varepsilon)$ of width ε . Moreover, for $k = 0, 1, 2, \dots$ one has the exterior estimate*

$$\|u_\lambda\|_{C^k(A_j^+(\varepsilon))} \leq C_{k,\varepsilon} \lambda^{k+1} e^{\lambda\varepsilon} \|\varphi_\lambda\|_{L^2(\partial D_j)}.$$

Proof. From Proposition 2.3.4, we know $P = M + R$ where $R \in \Psi_a^{-\infty}(\partial D)$. We use this characterization of P to estimate Fourier coefficients of $\varphi_\lambda|_{\partial D_j}$ along the boundary circle ∂D_j . To simplify the writing in the following we assume that $L_j = 2\pi$ and that the center of the disc D_j is the origin. Also, we abuse notation somewhat and simply write $\varphi_\lambda(s)$ for $\varphi_\lambda(q_j(s))$ in (2.4.2)-(2.4.4) below.

Then, for $k \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned} \hat{\varphi}_\lambda(k) &= \langle e^{isk}, \varphi_\lambda \rangle = \frac{1}{|k|} \langle M e^{isk}, \varphi_\lambda \rangle = \frac{1}{|k|} \langle e^{isk}, M \varphi_\lambda \rangle = \frac{1}{|k|} \langle e^{isk}, (P - R) \varphi_\lambda \rangle \\ &= \frac{\lambda}{|k|} \langle e^{isk}, \varphi_\lambda \rangle + \frac{1}{|k|} O(e^{-\tau|k|}) \|\varphi_\lambda\| = \frac{\lambda}{|k|} \hat{\varphi}_\lambda(k) + \frac{1}{|k|} O(e^{-\tau|k|}) \|\varphi_\lambda\|. \end{aligned} \quad (2.4.2)$$

From (2.4.2), we observe

$$(|k| - \lambda) \hat{\varphi}_\lambda(k) = O(e^{-\tau|k|}), \quad (2.4.3)$$

and hence,

$$\hat{\varphi}_\lambda(k) = O(e^{-\tau|k|}), \quad |k| \geq \lambda + 1. \quad (2.4.4)$$

It follows that φ_λ holomorphically continues to the strip $[0, 2\pi] + i[-\varepsilon, \varepsilon]$ and so $\varphi_\lambda|_{\partial D_j}$ holomorphically continues to an annular neighbourhood of ∂D_j of any width $\varepsilon < \tau$. In terms of the parametrizing coordinates $s \in [0, 2\pi]$, one holomorphically continues φ_λ to the annulus

$$A_j(\varepsilon) := \{x \in \mathbb{C}; x = e^{iw}, w \in [0, 2\pi] + i[-\varepsilon, \varepsilon]\}.$$

Without loss of generality, we assume here that the set where $\text{Im } w < 0$ (denoted by $A_j^-(\varepsilon)$) is the part of the annulus lying inside the domain D and $\text{Im } w > 0$ (denoted by $A_j^+(\varepsilon)$) is the part lying outside. Note that $\varepsilon > 0$ is *independent* of λ . It also follows from (2.4.3) that the holomorphic continuation, $\varphi_\lambda^\mathbb{C}$ of the boundary Steklov eigenfunction φ_λ satisfies

$$|\varphi_\lambda^\mathbb{C}(x)| \leq C_\varepsilon e^{\lambda|\text{Im } w|} \|\varphi_\lambda\|_{L^2(\partial D_j)}, \quad x = e^{iw}, \quad (2.4.5)$$

with an appropriate constant $C_\varepsilon > 0$.

Now we need to construct the harmonic continuation of the interior Steklov eigenfunction $u_\lambda \in C^\omega(D)$. By Green's formula,

$$u_\lambda(x) = (\mathcal{D}l\varphi_\lambda)(x) - \lambda(\mathcal{S}lg\varphi_\lambda)(x), \quad x \in D. \quad (2.4.6)$$

Since the harmonic continuation of both terms in (2.4.6) is carried out in the same way, we consider here the second term and then just indicate the minor changes necessary to deal with the first one. Since the multiplicative factor λ is irrelevant to the continuation, we just consider

$$(\mathcal{S}lg\varphi_\lambda)(x) = \frac{1}{2\pi} \int_{\partial D_j} \log|x - q| g(q) \varphi_\lambda(q) dq + R\varphi_\lambda(x), \quad x \in A_j^-(\delta) \quad (2.4.7)$$

where $R\varphi_\lambda = \frac{1}{2\pi} \int_{\partial D \setminus \partial D_j} \log|x - q| g(q) \varphi_\lambda(q) dq$. That $R\varphi_\lambda(q)$ harmonically continues to $x \in A_j^+(\varepsilon)$ is immediate since the kernels $G(x, q)$ with $(x, q) \in A_j(\varepsilon) \times \partial D_k, j \neq k$ are supported off-diagonal and so $G(x, q)$ is harmonic in each variable. Thus, it is enough to analyze the first term on the RHS of (2.4.7).

The arclength parametrization of ∂D_j is given by $q(s) = e^{is} \in \partial D_j, s \in [0, 2\pi]$. We let $\Theta_\lambda(s) := \int_0^s g(q(s')) \varphi_\lambda(q(s')) dq(s')$. First, by application of Green's formula in (2.1.1) with harmonic measure $\delta_{\partial D_j}$, it follows that $\int_0^{2\pi} g(q(s')) \varphi_\lambda(q(s')) dq(s') = 0$ and so, in particular, $\Theta_\lambda \in C^\omega([0, 2\pi])$. Moreover, since

$$|\widehat{\Theta_\lambda(k)}| \lesssim |k|^{-1} |\widehat{g\varphi_\lambda(k)}|,$$

it has the same holomorphic continuation properties as $g\varphi_\lambda(s)$ (with continuation to the strip $[0, 2\pi] + i[-\varepsilon, \varepsilon]$). Then, making the definition

$$\Phi_\lambda(e^{is}) := \Theta_\lambda(s), \quad s \in [0, 2\pi],$$

it follows that $\Phi_\lambda \in C^\omega(\partial D_j)$ holomorphically continues to the ε -annulus $A_j(\varepsilon)$ and satisfies (2.4.5) there. Then the RHS of (2.4.7) can be written in the form

$$-\frac{1}{2\pi} \int_0^{2\pi} \log|x - e^{is}| \partial_s(\Phi_\lambda^\mathbb{C}(x) - \Phi_\lambda(e^{is})) ds + R\varphi_\lambda(x).$$

Here, we continue to view $x = e^{iw} \in \mathbb{C}$ as a complex variable. Then, writing $2 \log|x - e^{is}| = \log(x - e^{is}) + \log(\bar{x} - e^{-is})$, an integration by parts with respect to ∂_s gives

$$\mathcal{S}l(g\varphi_\lambda)(x) = \frac{1}{2\pi} \text{Re} \int_0^{2\pi} \frac{\Phi_\lambda^\mathbb{C}(x) - \Phi_\lambda(e^{is})}{x - e^{is}} i e^{is} ds + R\varphi_\lambda(x); \quad x \in A_j^-(\varepsilon). \quad (2.4.8)$$

By Taylor expansion and Morera's theorem, for each $s \in [0, 2\pi]$, the function $f_{\lambda,s}(x) := \frac{\Phi_\lambda^\mathbb{C}(x) - \Phi_\lambda(e^{is})}{x - e^{is}}$ has the same analyticity properties as $\varphi_\lambda^\mathbb{C}(x)$ and consequently, $f_{\lambda,s}$ extends to a holomorphic function for all $x \in A_j(\varepsilon)$. Thus, the first term on the RHS extends harmonically to $x \in A_j(\varepsilon)$.

Also, we note that for each $x \in A_j(\varepsilon)$ the function $F_{\lambda,x}(s) = \frac{\Phi_\lambda^{\mathbb{C}}(x) - \Phi_\lambda(e^{is})}{x - e^{is}}$ extends holomorphically to $s \in [0, 2\pi] + i[-\varepsilon, \varepsilon]$. By Cauchy's formula, we can therefore deform the contour of integration into $A_j^-(\varepsilon)$ by letting $s \mapsto s - i\delta_0$ for any $0 < \delta_0 < \varepsilon$. Undoing the integration by parts in (2.4.8) then implies that we can write the harmonic continuation formula for the single layer term in the form:

$$\mathcal{S}l(g\varphi_\lambda)(x) = \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} \log |x - e^{is+\delta_0}| g(s - i\delta_0) \varphi_\lambda^{\mathbb{C}}(e^{is+\delta_0}) ds + R\varphi_\lambda(x), \quad x \in A_j^+(\varepsilon), \quad (2.4.9)$$

where \log in (2.4.9) denotes the principal logarithm (we note that $|x - e^{is}e^{\delta_0}| > 0$ when $x \in A_j^+(\varepsilon)$). The formula (2.4.9) is useful for estimating the harmonic continuation of $\mathcal{S}l(g\varphi)$ into the exterior of D (where it can blow-up exponentially in λ). Indeed, since $|\log(x - e^{is+\delta_0})| = O(1)$ uniformly for $(x, s) \in A_j^+(\delta_0/2) \times [0, 2\pi]$, it follows from (2.4.9) and (2.4.5) that

$$|\mathcal{S}l(g\varphi_\lambda)(x)| \lesssim e^{\lambda\delta_0} \|\varphi_\lambda\|_{L^2(\partial D_j)}, \quad (2.4.10)$$

uniformly for $x \in A_j^+(\delta_0/2)$.

The double layer term $\mathcal{D}l(\varphi_\lambda)(x)$ is analyzed similarly. In this case, we again rewrite the integral in complex form. Given the parametrization $[0, 2\pi] \ni s \mapsto e^{is} = q(s)$ of ∂D_j , the unit normal is $\nu(q(s)) = e^{is}$ and so,

$$\partial_{\nu(q(s))} G(x, q(s)) = \partial_{\nu(q(s))} \log |x - q(s)| = \operatorname{Re} \left(\frac{e^{is}}{x - e^{is}} \right).$$

As a result, we can write in complex form:

$$\mathcal{D}l(\varphi_\lambda)(x) = \operatorname{Re} \int_0^{2\pi} \varphi_\lambda(q(s)) \cdot \left(\frac{q(s)}{x - q(s)} \right) ds. \quad (2.4.11)$$

Next, writing $\frac{-1}{q'(s)} \partial_s \log(x - q(s)) = \frac{1}{x - q(s)}$, one integrates by parts in s with the effect of replacing the $O(|x - q(s)|^{-1})$ singularity in (2.4.11) with a $\log |x - q(s)|$ -type singularity. In the process, one creates an extra factor of λ coming from differentiation of $\varphi_\lambda(q(s))$. To see this, we note that from Proposition 2.3.4, $P = \oplus_j M_j + R$ with $R \in \Psi_a^{-\infty}$ and $P\varphi_\lambda = \lambda\varphi_\lambda$. Since $M_j(e^{ims_j}) = |m|e^{ms_j}$, for all m , it follows from Fourier expansion that

$$\|\partial_s \varphi_\lambda\|_{L^2(\partial D_j)} = \|M_j \varphi_\lambda\|_{L^2(\partial D_j)} = \|(P - R)\varphi_\lambda\|_{L^2(\partial D_j)} = O(\lambda) \|\varphi_\lambda\|_{L^2(\partial D_j)}.$$

So, modulo replacing $\varphi_\lambda(q(s))$ with $\partial_s \varphi_\lambda(q(s))$, the analysis proceeds as in the single-layer case and $|\mathcal{D}l(\varphi_\lambda)(x)|$ also satisfies the bound

$$|\mathcal{D}l(\varphi_\lambda)(x)| \lesssim \lambda e^{\lambda\delta_0} \|\varphi_\lambda\|_{L^2(\partial D_j)}, \quad (2.4.12)$$

uniformly for $x = e^{iw} \in A_j^+(\delta_0/2)$.

Finally, we note that for $k \geq 1$, $\|\mathcal{S}l(g\varphi_\lambda)\|_{C^k}$ and $\|\mathcal{D}l(\varphi_\lambda)\|_{C^k}$ are estimated in the same way, except that one must first integrate by parts k times in s to compensate for the k -derivatives in x . This creates an extra multiplicative factor of λ^k . As a consequence of (2.4.6)-(2.4.12), we have proved Proposition 2.4.1. \square

3. APPROXIMATION OF STEKLOV EIGENFUNCTIONS

In this section, we construct approximations to our Steklov eigenfunctions, which we will use in our study of their nodal sets. First we use Proposition 2.3.4 and our functions $e_{m,j}$ to construct these approximations on the boundary. Then we show how to construct quasimodes for Steklov eigenfunctions in the interior.

3.1. Approximation of boundary eigenfunctions. As a first step, Proposition 2.3.4 allows us to show that the boundary Steklov eigenfunctions φ_{λ_n} can be approximated up to exponentially decaying error by linear combinations of the quasimodes $e_{m,j}$. More specifically, one has the following

LEMMA 3.1.1. *There exist functions ψ_{λ_n} and f_{λ_n} in $C^\infty(\partial D)$, $n = 0, 1, 2, \dots$, such that $\varphi_{\lambda_n} = \psi_{\lambda_n} + f_{\lambda_n}$ for each n , and furthermore:*

- (1) *Let $\psi_{\lambda_n,j} := \psi_{\lambda_n}|_{\partial D_j}$, and suppose n is sufficiently large. Then for each boundary component ∂D_j , either $\psi_{\lambda_n,j} = 0$ or there exist $m_{n,j} \in \mathbb{Z}$ as well as $b_{n,j,+}$ and $b_{n,j,-} \in \mathbb{R}$ such that*

$$\psi_{\lambda_n,j} = b_{n,j,+} e^{2\pi i m_{n,j} s_j / L_j} + b_{n,j,-} e^{-2\pi i m_{n,j} s_j / L_j},$$

with

$$\left| \lambda_n - \frac{2\pi m_{n,j}}{L_j} \right| \leq \frac{2\pi}{L}, \quad L := \max\{L_1, \dots, L_k\}.$$

- (2) *For each $\ell \in \mathbb{Z}^+$, there exist constants $C_\ell > 0$ and $\tau_0 > 0$ such that*

$$\|f_{\lambda_n}\|_{C^\ell(\partial D)} \leq C_\ell \lambda_n^\ell e^{-\tau_0 \lambda_n}.$$

Remark 3.1.2. Since φ_{λ_n} are L^2 -normalized, for sufficiently large n , there exist boundary components for which $\psi_{\lambda_n,j}$ are not too small - in particular, large compared to $|f_{\lambda_n}|_{\partial D_j}$. These boundary components will be called *dominant* for u_{λ_n} (the precise definition follows in section 4.1). Those components on which $\psi_{\lambda_n,j}$ are small, or vanish identically, will be called *residual*.

Another consequence of Proposition 2.3.4 is eigenvalue asymptotics with exponentially decaying error:

PROPOSITION 3.1.3. *There exist $C > 0$ and $\tau_1 > 0$ such that $|\lambda_n - \mu_n| \leq C e^{-\tau_1 n}$.*

The proofs of Lemma 3.1.1 and Proposition 3.1.3 are relatively standard, but involve some technical linear algebra and are thus deferred to the Appendix.

Remark 3.1.4. From Proposition 3.1.3, for each eigenvalue λ_n with n sufficiently large, there exists at least one boundary component ∂D_j and an integer $m_{n,j}$ for which

$$\left| \lambda_n - \frac{2\pi m_{n,j}}{L_j} \right| = O(e^{-\tau_1 n}).$$

3.2. Interior quasimodes. In this section, we extend our approximate boundary eigenfunctions ψ_{λ_n} to the interior and show that the extensions are good approximations to the interior Steklov eigenfunctions. Recall that

$$\psi_{\lambda_n} = \sum_{j=1}^k \psi_{\lambda_n,j} = \sum_{j=1}^k (b_{n,j,+} e^{2\pi i m_{n,j} s_j / L_j} + b_{n,j,-} e^{-2\pi i m_{n,j} s_j / L_j}).$$

We will extend each $\psi_{\lambda_n, j}$ to an almost harmonic function $\bar{u}_{n, j} \in C^\infty(\bar{D})$, then sum them to create our global interior quasimode \bar{u}_n .

As a first step, we define the rate constant

$$\tau := \min_{j=1, \dots, k} \left\{ \tau_0, \frac{2\pi}{L_j} \tilde{\tau} \right\}, \quad (3.2.1)$$

where τ_0 is the exponential rate for the approximation in Lemma 3.1.1 and $\tilde{\tau}$ is the minimal analytic modulus in Proposition 2.3.4.

To construct the needed approximately harmonic functions, let $s_j : \partial D_j \rightarrow [0, L_j]$ be the arc length function (with metric g) along the boundary circle ∂D_j . Without loss of generality, we assume here that the circle ∂D_j is centered at the origin. Identifying ∂D_j with $\mathbb{R}/2\pi\mathbb{Z}$ via the parametrization $\theta : [0, 2\pi] \rightarrow \partial D_j$, we let

$$\Gamma_j := \min_{\theta \in [0, 2\pi]} s'_j(\theta).$$

Since s_j is monotone increasing, each of the constants $\Gamma_j > 0$. We let $s_j^{\mathbb{C}}(z)$ be the holomorphic continuation to the strip $S_\tau := [0, 2\pi] + i(-\tau, \tau)$ and put $N_j := \max_{\theta + i\xi \in S_\tau} |\partial_\theta^3 \text{Im } s_j^{\mathbb{C}}(\theta + i\xi)|$. We now fix a constant $\delta > 0$ once and for all:

$$\delta = \delta(\tau) := \min_{j=1, \dots, k} \{ \Gamma_j \tau / 2 - N_j \tau^3, \tau / 2 \}, \quad (3.2.2)$$

with $\tau > 0$ defined by (3.2.1). By possibly shrinking $\tau > 0$ further, we can (and do) assume from now on that $\delta > 0$.

In the following, we identify holomorphic extensions of 2π -periodic C^ω -functions to a strip over $[0, 2\pi]$ with an annular neighbourhood of $|z| = 1$ via the conformal map $z \mapsto e^{iz \frac{2\pi}{L_j}}$, $z \in [0, 2\pi] + i(-\tau, \tau)$. For each j , let

$$e_{n, j}^{\mathbb{C}}(z) := e^{\frac{2\pi}{L_j} i m_{n, j} s_j^{\mathbb{C}}(z)}, \quad z \in [0, 2\pi] + i(-\tau, \tau).$$

Then, natural harmonic extensions to the strip are given by

$$u_{n, j, +}(z) := \text{Re } e_{n, j}^{\mathbb{C}}(z), \quad u_{n, j, -}(z) = \text{Im } e_{n, j}^{\mathbb{C}}(z), \quad z \in [0, 2\pi] + i(-\tau, \tau). \quad (3.2.3)$$

With $z = \theta + i\xi$, we can write

$$\begin{aligned} u_{n, j, +}(z) &= e^{-\frac{2\pi}{L_j} m_{n, j} \text{Im } s_j^{\mathbb{C}}(z)} \cos \left(\frac{2\pi}{L_j} m_{n, j} \text{Re } s_j^{\mathbb{C}}(z) \right), \\ u_{n, j, -}(z) &= e^{-\frac{2\pi}{L_j} m_{n, j} \text{Im } s_j^{\mathbb{C}}(z)} \sin \left(\frac{2\pi}{L_j} m_{n, j} \text{Re } s_j^{\mathbb{C}}(z) \right), \end{aligned}$$

where, by Taylor expansion, $\text{Re } s_j^{\mathbb{C}}$ and $\text{Im } s_j^{\mathbb{C}}$ are real-analytic functions with

$$\begin{aligned} \text{Im } s_j^{\mathbb{C}}(\theta, \xi) &= s'(\theta)\xi + O(\xi^3), \\ \text{Re } s_j^{\mathbb{C}}(\theta, \xi) &= s(\theta) + O(\xi^2). \end{aligned} \quad (3.2.4)$$

As above, since the arclength function $s(\theta)$ is strictly increasing,

$$s'(\theta) \geq \Gamma_j > 0. \quad (3.2.5)$$

To define the global quasimode corresponding to u_{λ_n} we let $\chi_j \in C_0^\infty(\mathbb{R}^2; [0, 1])$ be standard cutoff equal to 1 in annular neighbourhood of ∂D_j of width $\tau/2$ and vanishing outside an

annulus of width τ . Then, with $x = e^{iz}$, and $j \in \{1, \dots, k\}$ corresponding to an *outer* boundary component and with $b_{n,j,\pm}$ as in Lemma 3.1.1, we define the global interior quasimode to be

$$\begin{aligned} \bar{u}_{n,j}(x) &:= \chi_j(x) \cdot (b_{n,j,+} u_{n,j,+}(z) + b_{n,j,-} u_{n,j,-}(z)) \\ &= \chi_j(x) \cdot \left(b_{n,j,+} e^{-\frac{2\pi}{L_j} m_{n,j} \operatorname{Im} s_j^{\mathbb{C}}(z)} \cos\left(\frac{2\pi}{L_j} m_{n,j} \operatorname{Re} s_j^{\mathbb{C}}(z)\right) \right. \\ &\quad \left. + b_{n,j,-} e^{-\frac{2\pi}{L_j} m_{n,j} \operatorname{Im} s_j^{\mathbb{C}}(z)} \sin\left(\frac{2\pi}{L_j} m_{n,j} \operatorname{Re} s_j^{\mathbb{C}}(z)\right) \right). \end{aligned} \quad (3.2.6)$$

We note that in the case of an outer boundary component, the point $x = e^{iz}$ with $\operatorname{Im} z = \xi > 0$ will lie in an interior collar neighbourhood of the boundary circle. In view of (3.2.4), for such points, $\operatorname{Im} s_j^{\mathbb{C}}(z) > 0$ and so, the quasimode (3.2.6) decays exponentially in n .

When ∂D_j with $j \in \{1, \dots, k\}$ is an *inner* boundary component, we replace $e_{n,j}^{\mathbb{C}}(z)$ above with $\tilde{e}_{n,j}^{\mathbb{C}}(z) := e^{-\frac{2\pi}{L_j} i m_{n,j} s_j^{\mathbb{C}}(z)}$ and replace $u_{n,j,\pm}(z)$ in (3.2.6) with $\operatorname{Re} \tilde{e}_{n,j}^{\mathbb{C}}(z)$ and $\operatorname{Im} \tilde{e}_{n,j}^{\mathbb{C}}(z)$ respectively. In this case, the corresponding quasimodes decay exponentially in an interior collar neighbourhood where $\operatorname{Im} s_j^{\mathbb{C}}(z) < 0$.

The function $\bar{u}_{n,j} \in C^\infty(\bar{D})$ is approximately harmonic in D and agrees with the boundary quasimode $\psi_{\lambda_n,j}$ on ∂D_j . Indeed, since $\Delta u_{n,j,\pm} = 0$, by Leibniz' rule for the Laplacian

$$\begin{aligned} \Delta \bar{u}_{n,j} &= \Delta[\chi_j \cdot (b_{n,j,+} u_{n,j,+} + b_{n,j,-} u_{n,j,-})] \\ &= 2\nabla_x \chi_j \cdot \nabla(b_{n,j,+} u_{n,j,+} + b_{n,j,-} u_{n,j,-}) + \Delta \chi_j (b_{n,j,+} u_{n,j,+} + b_{n,j,-} u_{n,j,-}). \end{aligned}$$

Since the derivatives of χ_j are supported in an annular neighbourhood of ∂D_j where $\tau/2 < |x| < \tau$, it follows in view of (3.2.4) that with $\delta = \delta(\tau)$ as in (3.2.2),

$$\|\Delta_x \bar{u}_{n,j}\|_{C^0(\bar{D})} = O(\lambda_n e^{-\delta \lambda_n})$$

and similarly,

$$\|\Delta_x \bar{u}_{n,j}\|_{C^\ell(\bar{D})} = O_\ell(\lambda_n^{\ell+1} e^{-\delta \lambda_n}), \quad \ell \in \mathbb{Z}^+. \quad (3.2.7)$$

Now for each n , we define the *interior quasimode*

$$\bar{u}_n := \sum_{j=1}^k \bar{u}_{n,j}. \quad (3.2.8)$$

As we now prove, the function \bar{u}_n in (3.2.8) is the quasimode approximation to the Steklov eigenfunction u_{λ_n} in a collar neighbourhood of the boundary ∂D that we seek.

LEMMA 3.2.9. *For any fixed $\ell \in \mathbb{Z}^+$, there exist constants $C_j(\ell) > 0$; $j = 1, 2$, such that*

$$\|u_{\lambda_n} - \bar{u}_n\|_{C^\ell(\bar{D})} \leq C_1(\ell) \lambda_n^{C_2(\ell)} e^{-\delta \lambda_n}.$$

Proof. For each n , let

$$\bar{f}_n := u_{\lambda_n} - \bar{u}_n, \quad (3.2.10)$$

so that \bar{f}_n is the error in the interior quasimode approximation. To $\|\bar{f}_n\|_{C^\ell(\bar{D})}$ we note that by construction, $\bar{u}_n = \psi_{\lambda_n}$ on ∂D , so $\bar{f}_n|_{\partial D} = f_{\lambda_n}$, which by Lemma 3.1.1 and the definition of δ shows that

$$\|\bar{f}_n|_{\partial D}\|_{C^\ell(\partial D)} \leq C_k \lambda_n^\ell e^{-\tau_0 \lambda_n} \leq C_\ell \lambda_n^\ell e^{-\delta \lambda_n}. \quad (3.2.11)$$

Moreover, both \bar{u}_n and \bar{f}_n are approximately harmonic. Indeed, in view of (3.2.7) and (3.2.8), we see

$$\|\Delta_x \bar{f}_n\|_{C^\ell(\bar{D})} = \|\Delta_x \bar{u}_n\|_{C^\ell(\bar{D})} \leq C_\ell \lambda_n^{\ell+1} e^{-\delta \lambda_n}. \quad (3.2.12)$$

To begin, we claim that

$$\|\bar{f}_n\|_{C^0(\bar{D})} \leq C \lambda_n e^{-\delta \lambda_n}. \quad (3.2.13)$$

Indeed, let Φ be the solution of $\Delta \Phi = -1$ on D satisfying the Dirichlet boundary condition $\Phi|_{\partial D} = 0$. Then, by the maximum principle, $\Phi > 0$ in D . We define

$$\bar{g}_n := C_0 e^{-\delta \lambda_n} + C_1 \lambda_n e^{-\delta \lambda_n} \Phi > 0.$$

By (3.2.11) and (3.2.12), $\bar{f}_n \leq \bar{g}_n$ on ∂D with $\Delta \bar{f}_n \geq \Delta \bar{g}_n$. Similarly, $\bar{f}_n \geq -\bar{g}_n$ on ∂D with $\Delta(-\bar{g}_n) \geq \Delta \bar{f}_n$ on D . Consequently, again by the maximum principle,

$$|\bar{f}_n| \leq |\bar{g}_n| = \bar{g}_n \text{ on } \bar{D},$$

which proves (3.2.13).

To treat the cases where $m \geq 1$, we note that by standard elliptic estimates ([Kr], Theorem 5.4.1), for any $m \in \mathbb{Z}^+$ even,

$$\|\bar{f}_n\|_{H^m(\bar{D})} \leq C_m (\|\Delta^{\frac{m}{2}} \bar{f}_n\|_{L^2(\bar{D})} + \|\bar{f}_n\|_{L^2(\bar{D})} + \|\bar{f}_n|_{\partial D}\|_{H^{m-\frac{1}{2}}(\partial \bar{D})}). \quad (3.2.14)$$

Substitution of (3.2.13) and (3.2.11) in (3.2.15) gives

$$\|\bar{f}_n\|_{H^m(\bar{D})} \leq C_m (\|\Delta^{\frac{m}{2}} \bar{f}_n\|_{L^2(\bar{D})} + \lambda_n e^{-\delta \lambda_n} + \lambda_n^{m-\frac{1}{2}} e^{-\delta \lambda_n}). \quad (3.2.15)$$

Finally, using (3.2.12) and the fact that $\|\Delta^{m/2} \bar{f}_n\|_{L^2(\bar{D})} \lesssim \|\Delta \bar{f}_n\|_{C^{m/2-1}(\bar{D})}$, it follows that

$$\|\bar{f}_n\|_{H^m(\bar{D})} = O_m((\lambda_n^{\frac{m}{2}} + \lambda_n^{m-\frac{1}{2}} + \lambda_n) e^{-\delta \lambda_n}) = O_m(\max\{\lambda_n^{m-\frac{1}{2}}, \lambda_n\} e^{-\delta \lambda_n}).$$

The proof then follows by an application of the Sobolev lemma which gives

$$\|\bar{f}_n\|_{C^\ell(\bar{D})} \lesssim \|\bar{f}_n\|_{H^m(\bar{D})}, \quad m > \ell + 1.$$

□

4. PROOF OF THEOREM 1.2.1

We now prove Theorem 1.2.1; first we prove lower bounds on nodal length and then proceed to upper bounds. Recall that u_λ is the Steklov eigenfunction on Ω with eigenvalue λ , and $v_\lambda = f^* u_\lambda$ is the corresponding eigenfunction on D . Throughout, we denote the nodal length of a function u on a set A by $\mathcal{L}(u, A)$. Note that since the Steklov eigenfunction in the conformal model is $v_\lambda = f^* u_\lambda$, the ratio of $\mathcal{L}(u_\lambda, A)$ and $\mathcal{L}(v_\lambda, f^*(A))$ is bounded from above and below by geometric constants independent of λ and of $A \subset \Omega$; hence we may work with the conformal model D without creating problems for our estimates. To simplify the writing, we abuse notation somewhat and henceforth denote both the Steklov eigenfunction and its image in the conformal model simply by u_λ .

4.1. Dominant and residual boundary components. Our first step is to group the boundary components $\partial D_j; j = 1, \dots, k$, into *dominant* and *residual* categories. The idea here is that since the basic quasimodes approximate actual Steklov eigenfunctions to $O(e^{-\tau\lambda})$ in C^k norm (by Lemma 3.1.1), eigenfunctions that have L^2 -norm less than $e^{-\tau\lambda}$ along a boundary component have no meaningful quasimode approximations. We note that since $\lambda_n \approx n$ (see section 2.3), the bounds in Lemma 3.1.1 in terms of $e^{-\tau_0 n}$ are comparable to ones in terms of $e^{-\tau\lambda_n}$ by possibly adjusting the constant $\tau > 0$. We choose to work here with the latter. The simple example of the annulus (see Example 1.4) shows that L^2 -norms of Steklov eigenfunctions can indeed be exponentially small with $\|\varphi_\lambda\|_{L^2(\partial D_j)} \sim e^{-\lambda C}$ along certain boundary circles with some $C > 0$. Moreover, it is not clear in general that $C > 0$ will be smaller than the exponential rate $\tau > 0$ appearing in the quasimode approximation in Lemma 3.1.1.

DEFINITION 4.1.1. Recall the definition of δ from (3.2.2). The boundary component ∂D_j is said to be *dominant* for u_λ provided

$$\|u_\lambda\|_{L^2(\partial D_j)} \geq e^{-\delta\lambda/3}.$$

Otherwise, it is said to be *residual*.

We refer to Example 1.4 for an illustration of the concepts of dominant and residual boundary components. If a surface has a single boundary component, it is obviously dominant for each u_λ .

A key observation about our dominant and residual boundary components is the following. Recall the approximation in Lemma 3.1.1, where the approximate boundary eigenfunction ψ_{λ_n} is a linear combination of trigonometric polynomials on each boundary component ∂D_j with coefficients $b_{n,j,\pm}$. The following is an immediate consequence of our definition of dominant and residual, combined with the error bounds in Lemma 3.1.1:

PROPOSITION 4.1.2. *If ∂D_j is dominant for u_{λ_n} , then*

$$|b_{n,j,+}|^2 + |b_{n,j,-}|^2 \geq e^{-\frac{2\delta\lambda_n}{3}} + O(e^{-4\delta\lambda_n}) \geq \frac{1}{2}e^{-\frac{2\delta\lambda_n}{3}}$$

as long as n is sufficiently large.

Remark 4.1.3. We note that even though L^2 -mass can be exponentially small along boundary components, it has recently been proved by a Carleman argument applied to Steklov eigenfunctions on the boundary $\partial\Omega$ ([Zh1, Theorem 1]), that they satisfy quantitative unique continuation in any dimension. More precisely, there is a constant $C = C(\Omega) > 0$, such that

$$\|\varphi_\lambda\|_{L^2(\partial\Omega_j)} \geq e^{-C\lambda} \quad \forall j = 1, \dots, k. \quad (4.1.4)$$

where $\partial\Omega_j; j = 1, \dots, k$ denote disjoint subsets of $\partial\Omega$.

Note that as $\lambda \rightarrow \infty$, the dominant components of the boundary change depending on λ . However, there are only finitely many configurations of dominant and residual components, so by passing to subsequences we may consider each configuration separately. Therefore, without loss of generality, we assume that the boundary components $\partial D_j; j = 1, \dots, P$ are dominant and the remaining ones with $j = P+1, \dots, k$ are residual. Given the L^2 -normalization condition (2.1.2), we may also assume that $\|u_\lambda\|_{\partial D_1} \geq \frac{1}{2k}$ (at least one dominant boundary

component has this property for sufficiently large λ , and there are only finitely many configurations). In the following, we denote by $A_j(\alpha)$ an α -width annular neighbourhood of the dominant boundary component ∂D_j .

4.2. Nodal length near a dominant boundary component. We now obtain optimal upper and lower bounds for the nodal length of a Steklov eigenfunction u_λ near a dominant boundary component.

LEMMA 4.2.1. *Let ∂D_j be a dominant boundary component and let $\tau > 0$ and $\delta(\tau) > 0$ be defined as above (see 3.2.2 and 3.2.1). Then, there exists $\alpha = \alpha(\tau) > 0$ and geometric constants $C_j = C_j(\partial D, D); j = 1, 2$, such that, for any Steklov eigenfunction u_λ*

$$C_1 \lambda \leq \mathcal{L}(Z_{u_\lambda} \cap A_j(\alpha)) \leq C_2 \lambda.$$

Proof. We use the decomposition $u_{\lambda_n} =: \bar{u}_{n,j} + \bar{f}_{n,j}$, with C^k -bounds on $\bar{f}_{n,j}$ as in Lemma 3.2.9 (note that $\bar{f}_{n,j} = \bar{f}_n$ in $A_j(\alpha)$). Rescale all of these functions by multiplying by $C > 0$, where C is chosen so that $b_{n,j,+}^2 + b_{n,j,-}^2 = 1$. By Proposition 4.1.2, $C \leq 2e^{\frac{2\delta\lambda_n}{3}}$ for sufficiently large n . None of the zero sets change, and by our observations about the size of C , we now have

$$\|f_{n,j}\|_{C^k} = \mathcal{O}_k(e^{-\delta\lambda_n/2+\delta\lambda_n/3}) = \mathcal{O}_k(e^{-\delta\lambda_n/6}). \quad (4.2.2)$$

For simplicity, assume that ∂D_j is the outer boundary (a similar argument works for the inner boundaries). As in section 3.2, we use the coordinates $(\theta, \xi) \in [0, 2\pi] + i(0, \delta)$ in the δ -strip model of the annulus $A(\alpha)$ with complex variable $z = \theta + i\xi$. Here, $\xi > 0$ corresponds to the interior of D where eigenfunctions decay exponentially in λ_n . A direct computation with the quasimodes of (3.2.6) gives

$$\bar{u}_{n,j}^2 + \lambda_n^{-2} |\partial_\theta \bar{u}_{n,j}|^2 \gtrsim e^{-\frac{4\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}(\theta, \xi)}. \quad (4.2.3)$$

We recall from (3.2.4) that $\text{Im } s_j^{\mathbb{C}}(\theta, \xi) = s'_j(\theta)\xi + O(\xi^3)$ with $\tilde{\Gamma}_j \geq s'_j(\theta) \geq \Gamma_j > 0$ and so $\text{Im } s_j^{\mathbb{C}}(\theta, \xi) \approx \xi$ for $0 \leq \xi < C_1(\tau)$ with $C_1(\tau)$ small. Thus, choosing $C_1(\tau) > 0$ sufficiently small, one can arrange that

$$0 \leq \text{Im } s_j^{\mathbb{C}}(\theta, \xi) < 2C_1(\tau)\tilde{\Gamma}_j.$$

Then, by possibly shrinking $C_1(\tau) > 0$ in (4.2.3) further, one can arrange that for $0 \leq \xi \leq C_1(\tau)$,

$$\bar{u}_{n,j}^2 + \lambda_n^{-2} |\partial_\theta \bar{u}_{n,j}|^2 \gtrsim e^{-\delta m_{n,j}/5} \gtrsim e^{-\delta\lambda_n/5}. \quad (4.2.4)$$

In the last inequality in (4.2.4), we have used the fact that $|m_{n,j} - \lambda_n| < 2\pi/L$.

Now, fix $\alpha = C_1(\tau)$ and let E_{λ_n} be the subset of $A_j(\alpha) = \{(\theta, \xi); 0 \leq \xi < \alpha\}$ where $|\bar{u}_{n,j}|^2 \leq e^{-\delta\lambda_n/4}$. By formula (4.2.2), for λ_n sufficiently large, $Z_{u_{\lambda_n}} \cap A_j(\alpha) \subseteq E_{\lambda_n}$. But using (4.2.4), for λ_n sufficiently large,

$$|\partial_\theta \bar{u}_{n,j}(\theta, \xi)| \geq C\lambda_n e^{-\frac{2\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}(\theta, \xi)} \geq C'\lambda_n e^{-\delta\lambda_n/10}, \quad (\theta, \xi) \in E_{\lambda_n}. \quad (4.2.5)$$

Using (4.2.2), eigenfunctions are approximated by quasimodes in C^k -norm to $O(e^{-\delta\lambda_n/6})$ -error and so, it follows from (4.2.5) that for the actual eigenfunctions,

$$|\partial_\theta u_{\lambda_n}(\theta, \xi)| \geq C\lambda_n e^{-\frac{2\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}(\theta, \xi)} \geq C'\lambda_n e^{-\delta\lambda_n/10}, \quad (\theta, \xi) \in E_{\lambda_n}.$$

So on each connected component of E_{λ_n} , u_{λ_n} is either monotonically increasing or monotonically decreasing in θ . Since for each fixed ξ , $\bar{u}_{n,j}$ has exactly one zero in each connected component of E_{λ_n} , so must u_{λ_n} for each fixed ξ . From (4.2.5), it follows by the analytic implicit function theorem that the nodal set $Z_{u_{\lambda_n}} \cap A_j(\alpha)$ is a union of C^ω curves, one in each connected component of E_{λ_n} , which are graphs with dependent variable θ and independent variable ξ . There are precisely $2\mu_n$ zeroes of $\bar{u}_{n,j}$, so we can write $Z_{u_{\lambda_n}} \cap A_j(\alpha)$ as a union of graphs of C^ω functions $f_1(\xi), \dots, f_{2\mu_n}(\xi)$.

It remains to control the geometry of these graphs. To do this, we claim that there exists a constant $C' > 0$ such that

$$|\partial_\xi u_{\lambda_n}| \leq C' |\partial_\theta u_{\lambda_n}| \text{ in } E_{\lambda_n}. \quad (4.2.6)$$

Given (4.2.6), it follows by the chain rule that $|f'_i(\xi)|$ is uniformly bounded in λ_n for each ξ . Therefore the arc length of each $f_i(\xi)$ is uniformly bounded above and below. Since there are $2\mu_n$ such graphs, and $2\mu_n \sim 2\lambda_n$, the result follows.

To prove (4.2.6), we note that by (4.2.5),

$$|\partial_\theta \bar{u}_{n,j}(\theta, \xi)| \geq C_2 \lambda_n e^{-\frac{2\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}(\theta, \xi)}, \quad (\theta, \xi) \in E_{\lambda_n}.$$

We also note that from (3.2.4), $\partial_\xi \text{Re } s_j^{\mathbb{C}}(\theta, \xi) = O(\xi)$ where $|\xi| < \alpha(\tau) \ll 1$ in E_{λ_n} and $\partial_\xi \text{Im } s_j^{\mathbb{C}}(\theta, \xi) = s'_j(\theta) + O(\xi^2)$ with $s'_j(\theta) \geq \Gamma_j > 0$. Thus, by another direct computation with the quasimodes (3.2.6), it follows that

$$|\partial_\xi \bar{u}_{n,j}(\theta, \xi)| \leq C_2 \lambda_n e^{-\frac{2\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}(\theta, \xi)}, \quad (\theta, \xi) \in E_{\lambda_n}.$$

The analogue of (4.2.6) is therefore immediately true for the quasimodes $\bar{u}_{n,j}$ in place of the eigenfunctions u_{λ_n} . Transferring to u_{λ_n} via Lemma 3.2.9 introduces errors that are $\mathcal{O}(e^{-\delta\lambda_n/6})$. However, in view of (4.2.5), these are lower-order than $e^{-\frac{2\pi m_{n,j}}{L_j} \text{Im } s_j^{\mathbb{C}}} \gtrsim e^{-\lambda_n \delta/10}$ in E_{λ_n} . Such errors are therefore negligible, so the pointwise estimate in (4.2.6) is satisfied for the actual eigenfunctions u_{λ_n} . \square

This gives both upper and lower bounds for nodal length near dominant boundary components, and the overall lower bound is an immediate corollary. It remains to estimate, from above, nodal lengths in the interior and near residual boundary components.

4.3. Estimates for the harmonic extensions of interior eigenfunctions near residual boundary components. To estimate nodal lengths all the way up to residual boundary components, we need to extend our domain slightly near the residual boundary components (independent of λ) and use this slightly enlarged domain. In the following, we continue to work in the conformal model. So let \tilde{D} be an extension of D to include a (λ -independent) neighborhood of the residual boundaries. We claim the following:

LEMMA 4.3.1. *Given $\delta > 0$ as above, there exists an open domain $\tilde{D}(\delta) \supset \bar{D}$ with smooth boundary containing open neighbourhoods of the residual boundary circles such that the Steklov eigenfunctions φ_λ all have a harmonic extension to $\tilde{D}(\delta)$ denoted by u_λ . Moreover, in the extended neighbourhoods near the residual components one has the following estimate:*

$$\|u_\lambda\|_{C^1(\tilde{D}(\delta) \setminus \bar{D})} \leq C e^{-\frac{\delta}{6}\lambda}$$

for some geometric constant $C > 0$ depending on δ but independent of λ .

Proof. This follows from Proposition 2.4.1. Given any residual boundary component ∂D_j and $\delta_0 < \delta$, consider the annular neighbourhood $A_j(\delta_0) \supset \partial D_j$ with

$$\|u_\lambda\|_{C^1(A_j(\delta_0))} \leq C\lambda^2 e^{\lambda\delta_0} \|\varphi_\lambda\|_{L^2(\partial D_j)} = O(\lambda^2 e^{\lambda(-\frac{\delta}{3} + \delta_0)}).$$

Choose $\delta_0 < \frac{\delta}{6}$ and let $\tilde{D} = D \cup (\cup_{j=P+1}^k A_j(\delta_0))$. \square

To simplify notation, in the following we denote the extended domain $\tilde{D}(\delta)$ simply by \tilde{D} .

4.4. Nodal length bounds away from dominant boundary components. We begin the analysis away from the dominant boundary components by considering simply connected sets in the interior of the extended domain \tilde{D} . In this section, we prove nodal bounds on these sets, then complete the proof with a covering argument.

LEMMA 4.4.1. *Let $U \Subset \tilde{U}$ be simply connected open sets in \tilde{D} , each with C^∞ boundary, where:*

- (1) *U is not contained in $\cup_{j=1}^P A_j(\alpha)$, where $\alpha = \alpha(\tau)$ was fixed in section 4.2.*
- (2) *For each $j = 1, \dots, P$, $\partial\tilde{U} \cap \partial D_j$ is a nontrivial circular subarc of ∂D_j .*
- (3) *(Technical assumption) Translating the angle coordinate θ if necessary, we assume that $\partial\tilde{U} \cap A_j(\alpha)$, for each j , may be parametrized in the form $\{(\theta, \xi); |\theta| \leq \theta_0, \xi = F(\theta)\}$ with $F(\theta) = 0$ for $|\theta| < \varepsilon_0$, $F'(\theta) > 0$ for $\theta > \varepsilon_0$ and $F'(\theta) < 0$ for $\theta < -\varepsilon_0$.*

Then there exists a constant C , depending on U and \tilde{U} but independent of λ , such that

$$\mathcal{L}(u_\lambda, U) \leq C\lambda. \quad (4.4.2)$$

Remark 4.4.3. The technical assumption, made for convenience (see (4.4.8)), says that the boundary of \tilde{U} approaches each dominant boundary component ∂D_j in a monotone fashion, travels along it for some distance, then departs in a monotone fashion. Note that $F(\cdot)$ measures distance from the boundary ∂D_j . See Figure 1 for an illustration of sets U and \tilde{U} satisfying the hypotheses of the Lemma.

Proof. Step 1: Pick $x_0 \in U \setminus \cup_{j=1}^P A_j(\alpha)$, which is possible by condition (1). Conformally mapping \tilde{U} to a disk and x_0 to the origin keeps the functions harmonic, and changes the length of the nodal set by at most a geometric constant. Therefore, we may assume that \tilde{U} is a disk of radius 1 and that x_0 is the origin.

Note that U is contained in some disk B_r with $r < 1$. We cover B_r by finitely many disks A_1, \dots, A_k with centers x_1, \dots, x_k and radii r_1, \dots, r_k , where each A_i has the property that $4A_i$, the disk with center x_i and radius $4r_i$, is contained in B_1 . By finiteness, we only need to prove the bound for each A_i . By the nodal measure bound of Han and Lin [HL, Theorem 2.3.1],

$$\mathcal{L}(u_\lambda, A_i) \leq C_1 N_{u_\lambda}(2A_i), \text{ where } N_{u_\lambda}(2A_i) = \frac{2r_i \int_{2A_i} |\nabla u_\lambda|^2}{\int_{\partial(2A_i)} u_\lambda^2}. \quad (4.4.4)$$

Note that Han-Lin's result is stated for a ball of radius 1, but scaling shows that it also holds for a ball of radius r , modulo a geometric constant which we absorb into C_1 . Since $4A_i \subset B_1 = \tilde{U}$ for all $i = 1, \dots, k$, a uniform control estimate on the frequency function ([HL, Theorem 2.2.8], see also [NV, Section 3.2.2]) yields:

$$N_{u_\lambda}(2A_i) \leq C_2 N_{u_\lambda}(\tilde{U})$$

Then, summing up over the disks A_i and using Green's identity as well as the Cauchy-Schwarz inequality we get:

$$\mathcal{L}(u_\lambda, U) \leq \frac{\int_{\tilde{U}} |\nabla u_\lambda|^2}{\int_{\partial \tilde{U}} u_\lambda^2} \leq C \frac{\|\partial_\nu u_\lambda\|_{\partial \tilde{U}} \|u_\lambda\|_{\partial \tilde{U}}}{\|u_\lambda\|_{\partial \tilde{U}}^2}. \quad (4.4.5)$$

Here and further on we use the simplified notation $\|\cdot\|_{\tilde{U}} := \|\cdot\|_{L^2(\tilde{U})}$.

Step 2: Since the argument for each dominant circle is the same, consider here the intersection of the open set \tilde{U} with the τ -annular neighbourhood of one fixed dominant boundary circle, say ∂D_1 . Let us estimate the numerator from above and the denominator from below, beginning with the denominator. We continue to divide through by a constant and without loss of generality assume that $b_{n,1,+}^2 + b_{n,1,-}^2 = 1$, where $b_{n,1,+}$ and $b_{n,1,-}$ are constants defined in Lemma 3.1.1. Consider the norm over the portion of \tilde{U} consisting of an arc γ along ∂D_1 . By the interior quasimode approximation in Lemma 3.2.9, property (2) in Lemma 4.4.1 and a direct computation with the explicit quasimodes in (3.2.6), there is a constant $c' > 0$ such that

$$\|u_\lambda\|_{\partial \tilde{U}}^2 \geq \|\bar{u}_{n,1}\|_\gamma^2 - e^{-c'\lambda}, \quad (4.4.6)$$

where $\bar{u}_{n,1}(x)$ is the global interior quasimode defined in (3.2.6). Let the curve segment $\gamma = \{(\theta, \xi); |\theta| < \varepsilon_0, \xi = 0\}$ with some $\varepsilon_0 > 0$. Since $\text{Im } s_j^C(\theta, \xi = 0) = 0$, it follows from Lemma 3.2.9 that

$$\|\bar{u}_{n,1}\|_\gamma^2 = \int_{|\theta| < \varepsilon_0} \left| \frac{2\pi}{L_1} b_{n,1,+} \cos(m_{n,1} s(\theta)) + \frac{2\pi}{L_1} b_{n,1,-} \sin(m_{n,1} s(\theta)) \right|^2 d\theta \geq C(\varepsilon_0) > 0.$$

Together with (4.4.6) this yields a lower bound

$$\|u_\lambda\|_{\partial \tilde{U}}^2 \geq C'(\varepsilon_0) > 0. \quad (4.4.7)$$

To get upper bounds for the numerator in (4.4.5), we split the integral over $\partial \tilde{U}$ into three pieces: $\partial \tilde{U} = \gamma_1 \cup \gamma_2 \cup \gamma_3$. The first piece, $\gamma_1 = \partial \tilde{U} \cap A(\alpha)$ is the putative leading term, coming from the part of $\partial \tilde{U}$ inside the annulus $A(\alpha)$ around the dominant circle ∂D_1 . As a consequence of our technical assumption (3),

$$\begin{aligned} \|u_\lambda\|_{\gamma_1}^2 &= \int_{|\theta| < \varepsilon_0} \left| \frac{2\pi}{L_1} b_{n,1,+} \cos(m_{n,1} s(\theta)) + \frac{2\pi}{L_1} b_{n,1,-} \sin(m_{n,1} s(\theta)) \right|^2 d\theta \\ &\quad + \int_{\theta_0 > |\theta| > \varepsilon_0} \mathcal{O}(e^{-\Gamma_1 \lambda [F(\theta) + O(F(\theta)^3)]}) d\theta + O(e^{-\lambda \delta/2}) \approx 1, \end{aligned} \quad (4.4.8)$$

since $\int_{\theta_0 > |\theta| > \varepsilon_0} \mathcal{O}(e^{-\Gamma_1 \lambda [F(\theta) + O(F(\theta)^3)]}) d\theta = O(\lambda^{-1})$ for $\delta_0 > 0$ small by a change of variables $\theta \mapsto F(\theta)$, using that $|F'(\theta)| > 0$ in the range $\varepsilon_0 < |\theta| < \delta_0$.

The second piece $\gamma_2 = \partial \tilde{U} \cap [\bar{D} \setminus A(\alpha)]$. This is the piece of $\partial \tilde{U}$ in the closed domain \bar{D} outside the annulus. Here, we know by formula (4.2.2) that with $\alpha = C_1(\tau) > 0$ sufficiently small,

$$\|u_\lambda\|_{\gamma_2} = \|\bar{u}_{n,1}\|_{\gamma_2} + O(e^{-\frac{\delta\lambda}{6}}) = O(e^{-\frac{\Gamma_j \pi C_1(\tau) \lambda}{L_j}}) + O(e^{-\frac{\delta\lambda}{6}}).$$

Here, the first term is bounded using (3.2.6), (3.2.4) and (3.2.5).

Finally, the third piece $\gamma_3 = \partial\tilde{U} \setminus \bar{D}$ is the part of the boundary of \tilde{U} that is the exterior to \bar{D} in the extended domain \tilde{D} . Using Lemma 4.3.1 we have

$$\|u_\lambda\|_{\gamma_3} = O(e^{-\delta\lambda/6}).$$

Consequently,

$$\|u_\lambda\|_{\partial\tilde{U}} = O(1). \quad (4.4.9)$$

For the normal derivative term, the same decomposition argument shows that

$$\|\partial_\nu u_\lambda\|_{\partial\tilde{U}} = O(\lambda). \quad (4.4.10)$$

Then, from (4.4.7), (4.4.9) and (4.4.10) it follows that

$$\mathcal{L}(v_\lambda, U) \leq C \frac{\|\partial_\nu u_\lambda\|_{\partial\tilde{U}} \|u_\lambda\|_{\partial\tilde{U}}}{\|u_\lambda\|_{\partial\tilde{U}}^2} \leq C\lambda. \quad (4.4.11)$$

This completes the proof of the lemma. \square

Now we use a covering lemma to finish the proof.

LEMMA 4.4.12. *Suppose Ω is a connected, compact, smooth Riemannian manifold with smooth boundary $\partial\Omega$ consisting of k connected components M_1, \dots, M_k . Then there exists a covering of Ω by finitely many open sets, each of which is connected, contractible, and satisfies the assumptions on \tilde{U} in Lemma 4.4.1.*

Proof. First take a finite collection of open balls (and half-balls, near the boundary) $\{U_i\}$ which covers Ω (compactness guarantees finiteness is possible). Take small enough balls to ensure that \bar{U}_i is contractible for each i (for example, assume that each ball has radius less than the injectivity radius of Ω). For each i , we will find an open set V_i which contains U_i , has contractible closure, and moreover intersects each boundary component. The collection $\{V_i\}$ is then the desired cover.

So: begin with a set U_i . Since Ω is connected, for each j between 1 and k , let $\gamma_j : [0, 1] \rightarrow \Omega$ be a smooth path with $\gamma_j(0) \in U_i$ and $\gamma_j(1) \in M_j$.

First, let $t_1 = \sup\{t \in [0, 1] \mid \gamma_1(t) \in U_i\}$. Then let $\tilde{\gamma}_1$ be the portion of the image of γ_1 corresponding to $t \in [t_1, 1]$; let W_1 be open such that \bar{W}_1 is a contractible closed neighborhood of $\tilde{\gamma}_1$, and let $U_{i,1} = U_i \cup W_1$. Now $\bar{U}_{i,1}$ is contractible (as \bar{W}_1 contracts to $\tilde{\gamma}_1$, and $U_i \cup \tilde{\gamma}_1$ contracts to U_i), and intersects M_1 .

Then proceed inductively to define $U_{i,j}$; for each j , let $t_j = \sup\{t \in [0, 1] \mid \gamma_j(t) \in U_{i,j-1}\}$, let $\tilde{\gamma}_j$ be the portion of the image of γ_j corresponding to $t \in [t_j, 1]$, and let W_j be the interior of a contractible closed neighborhood of $\tilde{\gamma}_j$. Then let $U_{i,j} = U_{i,j-1} \cup W_j$. At each step, $\bar{U}_{i,j}$ is contractible, as it contracts to $\bar{U}_{i,j-1}$ which is assumed contractible by the inductive hypothesis. And $\bar{U}_{i,j}$ intersects M_j . Appropriately smoothing out and adjusting the boundary of $U_{i,k}$ in each neighborhood $A_j(\alpha)$ gives us a new set V_i satisfying all assumptions of Lemma 4.4.1, completing the proof. Note in particular that the technical assumption (3) in Lemma 4.4.1 is essentially local and can always be arranged. \square

Applying this lemma to \tilde{D} , we obtain a finite collection of open sets \tilde{U}_i , $i = 1, \dots, \ell$, which cover \tilde{D} . For each $x \in D \setminus \cup_{j=1}^P A_j(\alpha)$, which is an open subset compactly contained in the interior of \tilde{D} , there is an open ball U_x about x compactly contained in some \tilde{U}_i .

By compactness, a finite collection of these balls, $\{U_k\}_{k=1}^l$, cover $D \setminus \cup_{j=1}^P A_j(\alpha)$. Applying Lemma 4.4.1 to each such U_k and the \tilde{U}_i it is contained in gives us, for each k ,

$$\mathcal{L}(Z_{u_\lambda} \cap U_k) \leq C_k \lambda.$$

Finiteness of the covering completes the proof of Theorem 1.2.1.

4.5. Real analytic Riemannian surfaces: general case. We consider here the general case when (Ω, g) is a compact real-analytic Riemannian surface with boundary $\partial\Omega$, and show that the argument above applies to this setting. It is well known that given a real-analytic Riemannian manifold it is always possible to extend it across its boundary to a slightly larger open real-analytic Riemannian manifold (see, for instance, [EC, Example 5.50]). Let $(\tilde{\Omega}, \tilde{g})$ be such an extension of the surface (Ω, g) . On the open Riemann surface $\tilde{\Omega}$, by the Behnke-Stein Theorem ([V] Theorem 9.4.6), there exists a strictly-plurisubharmonic exhaustion function for $\tilde{\Omega}$, and so there is a Green's function $G(z, w) \in \mathcal{D}'(\tilde{\Omega} \times \tilde{\Omega})$ with the property that

$$\Delta_{g,z} G(z, w) = \delta(w - z), \quad (z, w) \in \tilde{\Omega} \times \tilde{\Omega},$$

and

$$G(z, w) = -\frac{1}{2\pi} \log |z - w| - a(z, w), \quad (4.5.1)$$

where a is harmonic in both variables and hence $a \in C^\omega(\tilde{\Omega} \times \tilde{\Omega})$.

Next, we note that although we have used the Koebe conformal model in the genus zero planar case for convenience, it is not necessary. Indeed, returning to the initial Steklov eigenfunctions $\varphi_\lambda \in C^\omega(\partial\Omega)$ one can directly derive the potential layer equations used to obtain the crucial quasimode approximations for the Steklov eigenfunctions along $\partial\Omega$. Therefore, the arguments of section 2.3 may be repeated with the only difference that one needs to take into account the contribution of $a(z, w)$. However, since this function is harmonic, its contribution can be absorbed in the last two terms of the expression in Claim 2.3.7. Consequently, just as in the planar case,

$$\frac{1}{2} \varphi_{\lambda_n} = \lambda_n G \varphi_{\lambda_n} - \lambda_n K_1 \varphi_{\lambda_n} - K_2 \varphi_{\lambda_n},$$

where the Schwartz kernels of K_1 and K_2 are elements of $C^\omega(\partial\Omega \times \partial\Omega)$ and G is the same as in Claim 2.3.7. The quasimode approximation results in Proposition 2.3.4 and Lemma 3.1.1 and the estimates in Lemmas 3.2.9 and 4.3.1 then follow as in the multiply-connected planar case after decomposing the boundary $\partial\Omega$ into dominant and residual components. This completes the proof in the general case. \square

APPENDIX A. PROOFS OF LEMMA 3.1.1 AND PROPOSITION 3.1.3

Throughout the appendix, for simplicity, we work with a real basis of eigenfunctions.

A.1. An auxiliary lemma. Lemma 3.1.1 depends on the presence of recurring spectral gaps, so in order to prove it, we must first prove the following lemma.

LEMMA A.1.1. *Fix any sufficiently small $\varepsilon > 0$. Then there exist pairwise disjoint closed intervals $\mathcal{I}_i = [A_i, B_i] \subset \mathbb{R}^+$, $i = 1, \dots, \infty$, with the following properties:*

- (1) $\sigma(P) \cup \sigma(M_j) \subseteq \bigcup_{i=1}^\infty \mathcal{I}_i$;
- (2) $A_{i+1} \geq B_i + \varepsilon$ for all i ;

- (3) For $i \geq 2$ and all j , each \mathcal{I}_i contains at most one distinct element of $\sigma(M_j)$;
 (4) For each i , there exists n such that $\lambda_n \in \mathcal{I}_i$ and $\mu_n \in \mathcal{I}_i$.

Proof of Lemma A.1.1. Pick $\varepsilon < \frac{\pi}{2kL}$, where L is the maximum of the boundary lengths L_1, \dots, L_k . Since $|\lambda_n - \mu_n| \rightarrow 0$, there exists N such that if $n \geq N$, then $|\lambda_n - \mu_n| < \varepsilon/2$.

Observe that since $\{\mu_n\}$ is a union of k arithmetic progressions, each with period $\geq 2\pi/L$, there are at most k elements of $\{\mu_n\}$ in any interval of length π/L . Thus any such interval must contain a gap of length at least $\pi/kL > 2\varepsilon$ with no elements of $\{\mu_n\}$. We may therefore choose some m_2 so that $m_2 > N$ and $\mu_{m_2} \geq \mu_{m_2-1} + 2\varepsilon$. Consider the interval $[\mu_{m_2}, \mu_{m_2} + \pi/L]$ and observe that it must itself contain a gap of length at least 2ε . Let $n_2 \geq m_2$ be such that μ_{n_2} is the left endpoint of the first such gap; we have $\mu_{n_2} - \mu_{m_2} \leq \pi/L$. Then let $m_3 = n_2 + 1$, so that μ_{m_3} is the right endpoint of that gap. Choosing n_3 as with n_2 and iterating this process, we produce $m_2, n_2, m_3, n_3, \dots$.

Now let

$$\mathcal{I}_1 = [0, \mu_{m_2} - 3\varepsilon/2]; \quad \mathcal{I}_i = [\mu_{m_i} - \varepsilon/2, \mu_{n_i} + \varepsilon/2] \quad \forall j \geq 2.$$

We claim that these intervals satisfy each property we want. Indeed, by construction, property (2) is automatic. Property (1) follows immediately from the fact that $|\lambda_n - \mu_n| < \varepsilon/2$ whenever $n \geq m_2 - 1 \geq N$. To see property (3), note that each \mathcal{I}_i with $i \geq 2$ has length at most $\pi/L + \varepsilon < 2\pi/L$, and hence contains at most one element of each arithmetic progression with period $\geq 2\pi/L$. Finally, property (4) is also immediate by construction (note that \mathcal{I}_1 contains $\mu_0 = 0$). This completes the proof. \square

We now use these intervals to split the sequence $\{\lambda_n\}$ into pieces with gaps of size at least ε between each. Specifically, fix some ε and let \mathcal{I}_i , $i = 1, 2, \dots$ be the intervals constructed in Lemma A.1.1. For each i , we let \mathcal{L}_i be the set of all j for which $\lambda_j \in \mathcal{I}_i$, and we say that

$$j_1 \sim j_2 \iff \exists i \in \mathbb{N} \text{ s.t. } j_1 \in \mathcal{I}_i \text{ and } j_2 \in \mathcal{I}_i. \quad (\text{A.1.2})$$

Since each interval \mathcal{I}_i for $i \geq 2$ contains at most k eigenvalues λ_n (and each interval, including \mathcal{I}_1 , contains at least one), there exists a universal constant C such that

$$j \in \mathcal{L}_i \Rightarrow j \leq Ci \text{ and } i \leq j. \quad (\text{A.1.3})$$

A.2. Completing the proofs.

Proof of Lemma 3.1.1. Let $\{\bar{e}_n\}$ be an orthonormal basis of eigenfunctions for M , with eigenvalues μ_n , and note that each $\bar{e}_{n,j} = \bar{e}_n|_{\partial D_j}$ is either a trigonometric polynomial with frequency μ_n or identically zero. Write $a_{j,k} = \langle \varphi_{\lambda_j}, \bar{e}_k \rangle$. By orthonormality and completeness of the eigenbases, we have

$$\bar{e}_k = \sum_{j=1}^{\infty} a_{j,k} \varphi_{\lambda_j}; \quad \varphi_{\lambda_k} = \sum_{j=1}^{\infty} a_{k,j} \bar{e}_j. \quad (\text{A.2.1})$$

We claim that if we set

$$\psi_{\lambda_n} = \sum_{j \sim n} a_{n,j} \bar{e}_j, \quad f_{\lambda_n} = \sum_{j \sim n} a_{n,j} \bar{e}_j, \quad (\text{A.2.2})$$

where \sim is defined by (A.1.2), then the conditions of the Lemma are satisfied. Indeed, condition (1) is obvious, since by the definition of our intervals \mathcal{I}_i , the frequency of each \bar{e}_j with $j \sim n$ is within $2\pi/L$ of λ_n . It remains to prove (2).

In what follows, we let C and $\tau > 0$ be universal constants (which may depend on the geometry of D and on the conformal map from D to Ω) and re-label at will. From Proposition 2.3.4 and the Weyl asymptotics, we know that there exists $\tau > 0$ such that $|P\bar{e}_j - \mu_j\bar{e}_j| \leq Ce^{-\tau j}$. Plugging in the first equation in (A.2.1), we see that

$$\left\| \sum_{n=1}^{\infty} a_{n,j}(\lambda_n - \mu_j)\varphi_{\lambda_n} \right\|_{\infty} \leq Ce^{-\tau j}$$

and hence that the same is true of the L^2 -norm. Summing only over the n with $n \asymp j$, we obtain

$$\left(\sum_{n \asymp j} a_{n,j}^2 (\lambda_n - \mu_j)^2 \right)^{1/2} \leq Ce^{-\tau j}. \quad (\text{A.2.3})$$

By property (2) of Lemma A.1.1, we know $|\lambda_n - \mu_j| \geq \varepsilon$ whenever $n \asymp j$, so

$$\sum_{n \asymp j} a_{n,j}^2 \leq Ce^{-\tau j}. \quad (\text{A.2.4})$$

However, by (A.2.1), for each j we have $\sum_n a_{n,j}^2 = 1$. So

$$1 - Ce^{-\tau j} \leq \sum_{n \asymp j} a_{n,j}^2 \leq 1. \quad (\text{A.2.5})$$

In addition, suppose $j_1 \sim j_2$ with $j_1 < j_2$. Then, again using (A.2.1) and the orthonormality of φ_k , as well as Cauchy-Schwarz,

$$\left| \sum_{n \asymp j_1} a_{n,j_1} a_{n,j_2} \right| = \left| - \sum_{n \asymp j_1} a_{n,j_1} a_{n,j_2} \right| \leq \left(\sum_{n \asymp j_1} a_{n,j_1}^2 \right)^{1/2} \left(\sum_{n \asymp j_1} a_{n,j_2}^2 \right)^{1/2} \leq Ce^{-\tau j_1}. \quad (\text{A.2.6})$$

Thinking of $A = \{a_{n,j}\}$ as an infinite matrix, let M_i be the square submatrix of A with n and j in \mathcal{L}_i ; note that by condition (3) of Lemma A.1.1 M_i is of size at most $2k \times 2k$. We interpret (A.2.5) and (A.2.6) as saying that the columns of M_i have almost unit length and are almost orthogonal to each other. In particular, for each i , we may write $M_i^T M_i = I + R_i$. By (A.2.5) and (A.2.6), there exists a constant $\tau > 0$ such that $\|R_i\|_{\infty} \leq Ce^{-\tau i}$ (the norm $\|M\|_{\infty}$ denotes the supremum of the entries of M). Therefore, for large enough i , $(I + R_i)$ is always invertible, with inverse of the form $I + S_i$, with $\|S_i\|_{\infty} \leq Ce^{-\tau i}$ as well. In fact, $(I + R_i)^{-1} M_i^T M_i = I$, so M_i^{-1} exists and equals $(I + R_i)^{-1} M_i^T = M_i^T + S_i M_i^T$. Multiply the equation $M_i^T M_i = I + R_i$ on the left by M_i and on the right by M_i^{-1} to yield

$$M_i M_i^T = M_i (I + R_i) M_i^{-1} = I + M_i R_i (I + S_i) M_i^T.$$

We see now that $M_i M_i^T$ has the form $I + Q_i$, where Q_i satisfies $\|Q_i\|_{\infty} \leq Ce^{-\tau i}$ (note that the entries of M_i and M_i^T are bounded by 1).

This now tells us that the *rows* of M_i have almost unit length: i.e. that

$$1 - Ce^{-\tau i} \leq \sum_{j \sim n, j \in \mathcal{L}_i} a_{n,j}^2 \leq 1.$$

By subtraction and (A.1.3), there exists $\tau > 0$ such that

$$\sum_{j \asymp n} a_{n,j}^2 \leq Ce^{-\tau n}, \quad (\text{A.2.7})$$

which is the analogue of (A.2.4), but summing in j instead of in n . Additionally, it is an immediate consequence of (A.2.4) and (A.2.7) that if $j \asymp n$,

$$a_{n,j}^2 \leq Ce^{-\tau j}; \quad a_{n,j}^2 \leq Ce^{-\tau n}. \quad (\text{A.2.8})$$

By the Sobolev embedding theorem, for each $k \in \mathbb{N}_0$, we have $\|\cdot\|_{C^k(\partial D)} \leq C\|\cdot\|_{H^{k+1}(\partial D)}$. We apply this to f_{λ_n} , knowing that we can compute Sobolev norms of the \bar{e}_j directly:

$$\|f_{\lambda_n}\|_{C^k}^2 \leq C\|f_n\|_{H^{k+1}}^2 \leq C \sum_{j \asymp n} (1 + \mu_j)^{k+1} |a_{n,j}|^2.$$

By Weyl asymptotics of the μ_j ,

$$\|f_{\lambda_n}\|_{C^k}^2 \leq C \sum_{j \asymp n} (1 + j)^{k+1} a_{n,j}^2 = C \sum_{j \asymp n, j < n} (1 + j)^{k+1} a_{n,j}^2 + C \sum_{j \asymp n, j > n} (1 + j)^{k+1} a_{n,j}^2.$$

Using (A.2.8), then the Weyl asymptotics again, we have

$$\begin{aligned} \|f_{\lambda_n}\|_{C^k}^2 &\leq C \sum_{j \asymp n, j < n} (1 + n)^{k+1} e^{-\tau n} + C \sum_{j \asymp n, j > n} (1 + j)^{k+1} e^{-\tau j} \\ &\leq Cn(1 + n)^{k+1} e^{-\tau n} + C \int_n^\infty (1 + t)^{k+1} e^{-\tau t} dt \leq C\lambda_n^{k+2} e^{-\tau n}. \end{aligned} \quad (\text{A.2.9})$$

Choosing τ slightly less than the current $\tau/2$, we can absorb λ_n^2 and take square roots, completing the proof. \square

Proof of Proposition 3.1.3. From the discussion before (A.2.3), considering each term individually, we see that for every n and j ,

$$|a_{n,j}(\lambda_n - \mu_j)| \leq Ce^{-\tau j}. \quad (\text{A.2.10})$$

Using the matrix notation from the previous proof, let V_i and W_i be the diagonal matrices whose entries are $\{\lambda_j : j \in \mathcal{L}_i\}$ and $\{\mu_j : j \in \mathcal{L}_i\}$ respectively. Using (A.1.3), (A.2.10) implies the statement

$$\|M_i V_i - W_i M_i\|_\infty \leq Ce^{-\tau i}.$$

For sufficiently large i , M_i is invertible with inverse uniformly bounded in the $\|\cdot\|_\infty$ matrix norm. We deduce that

$$M_i V_i M_i^{-1} = W_i + \bar{R}_i, \quad \|\bar{R}_i\|_\infty \leq Ce^{-\tau i}.$$

So $W_i + \bar{R}_i$ is diagonalized by M_i and has eigenvalues $\{\lambda_j : j \in \mathcal{L}_i\}$. By the Bauer-Fike theorem (see, e.g., [HJ, Observation 6.3.1]), the eigenvalues of $W_i = (W_i + \bar{R}_i) - \bar{R}_i$ lie in disks centered at each λ_j of radius $\|M_i\|_\infty \cdot \|M_i\|_\infty^{-1} \cdot \|\bar{R}_i\|_\infty \leq Ce^{-\tau i}$. Although the perturbed eigenvalues may move from one disk to another if there is overlap, there are at most k disks, so they can move at most $2kCe^{-\tau i}$. Relabeling $C = 2kC$, this shows that $|\lambda_n - \mu_n| \leq Ce^{-\tau i}$, where $\lambda_n \in \mathcal{L}_i$. The result follows immediately by another use of (A.1.3), replacing τ with $\tau_1 = \tau/C$. \square

REFERENCES

- [AM] G. Alessandrini and R. Magnanini, Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions, *SIAM J. Math. Anal.* **25** (1994), 1259–1268.
- [BL] K. Bellova and F. Lin, Nodal sets of Steklov eigenfunctions, *Calc. Var. and PDE* (2015), **54**(2), 2239–2268.
- [BK] L. Boutet de Monvel and P. Krée, Pseudodifferential operators and Gevrey classes, *Ann. l’Institut Fourier* **17** (1967) no.1, 295–323.
- [CT] Y. Canzani and J. Toth, Nodal sets of Schrödinger eigenfunctions in forbidden regions, *Ann. Henri Poincaré* (2016) 17: 3063. doi:10.1007/s00023-016-0488-3.
- [CM] T. H. Colding and W. P. Minicozzi II, Lower bounds for nodal sets of eigenfunctions, *Comm. Math. Phys.* **306** (2011), 777–784.
- [D] B. Dittmar, Sums of reciprocal Stekloff eigenvalues, *Math. Nachr.* **268** (2004), 44–49.
- [DF] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, *Invent. Math.* **93** (1988), 161–183.
- [Ed] J. Edward, An inverse spectral result for the Neumann operator on planar domains, *J. Funct. Anal.* **111** (1993), no. 2, 312–322.
- [EC] Y. Eliashberg and K. Cieliebak, From Stein to Weinstein and back: symplectic geometry of affine complex manifolds, *AMS Colloquium Publications* **59** (2012), American Mathematical Society, Providence, RI.
- [GPPS] A. Girouard, L. Parnowski, I. Polterovich, and D. Sher, The Steklov spectrum of surfaces: asymptotics and invariants, *Math. Proc. Camb. Phil. Soc.* **157** (2014), no. 3, 379–389.
- [GP] A. Girouard and I. Polterovich, Spectral geometry of the Steklov problem, arXiv:1411.6567, to appear in *J. Spec. Theory*.
- [HL] Q. Han and F.H. Lin, Nodal sets of solutions of elliptic PDE, Lecture notes available online at: <http://www3.nd.edu/~qhan/nodal.pdf>.
- [HZZ] B. Hanin, S. Zelditch and P. Zhou, Nodal sets of random eigenfunctions for the isotropic harmonic oscillator, *Int. Mat. Res. Not.* (2014), doi: 10.1093/imrn/rnu071.
- [HS] H. Hezari and C.D. Sogge, A natural lower bound for the size of nodal sets, *Anal. PDE.* **5** (2012), 1133–1137.
- [HW] H. Hezari and Z. Wang, Lower bounds for volumes of nodal sets: an improvement of a result of Sogge-Zelditch, *Proc. Symp. Pure Math.* **84** (2012), 229–235.
- [HL] P.D. Hislop and C.V. Lutzer, Spectral asymptotics for the Dirichlet-to-Neumann map for multiply-connected domains in \mathbb{R}^n , *Inverse Problems* **17** (2001), no. 6, 1717–1741.
- [HS] Z.-X. He and O. Schramm, Fixed points, Koebe uniformization and circle packings, *Annals of Math.* **137** (1993), 369–406.
- [HJ] R. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
- [KKP] M. Karpukhin, G. Kokarev and I. Polterovich, Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces, *Ann. Inst. Fourier* **64** (2014), no. 6, 2481–2502.
- [Kr] S. Krantz, *Partial Differential Equations and Complex Analysis*, Studies in Advanced Mathematics, CRC Press Inc., 1992.
- [KS] J.R. Kuttler and V.G. Sigillito, An inequality of a Stekloff eigenvalue by the method of defect, *Proc. Amer. Math. Soc.* **20** (1969), 357–360.
- [LM] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, vol. 1, Die Grundlehren der mathematischen Wissenschaften Vol. 181, Springer-Verlag, 1972.
- [M] D. Mangoubi, A remark on recent lower bounds for nodal sets, *Comm. PDE* **36** (2011), no. 12, 2208–2212.
- [NV] A. Naber and D. Valtorta, Volume estimates on the critical sets of solutions to elliptic PDE, arXiv:1403.4176.
- [MN] R. Morrey and L. Nirenberg, On the analyticity of the solutions of linear elliptic systems of partial differential equations, *Comm. Pure Appl. Math.* **10** (1957), 271–290.
- [Ro] G.V. Rozenblyum, On the asymptotics of the eigenvalues of certain two-dimensional spectral problems, *Sel. Math. Sov.* **5** (1986), 233–244.

- [Sh] S. E. Shamma, Asymptotic behavior of Steklov eigenvalues and eigenfunctions, *SIAM J. Appl. Math.* **20** (1971), 482–490.
- [SWZ] C. Sogge, X. Wang, and J. Zhu, Lower bounds for interior nodal sets of Steklov eigenfunctions, *Proc. Amer. Math. Soc.* **144** (2016), 4715–4722.
- [SZ] C.D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets, *Math. Res. Lett.* **18** (2011), 25–37.
- [Ta1] M. Taylor, *Partial differential equations II. Qualitative studies of linear equations*, Applied Mathematical Sciences **116** (1996), Springer-Verlag, New York.
- [Ta2] M. Taylor, Private communication.
- [Tr] F. Trèves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Plenum, New York, 1980.
- [TZ] J. Toth and S. Zelditch, Counting nodal lines which touch the boundary of an analytic domain, *J. Diff. Geom.* **81** (2009), 649–686.
- [V] D. Varolin, *Riemann surfaces by way of complex analytic geometry*, Graduate Studies in Mathematics Series **125**, Amer. Math. Soc, Providence, RI, 2011.
- [WZ] X. Wang and J. Zhu, A lower bound for the nodal sets of Steklov eigenfunctions, *Math. Res. Lett.* **22** (2015), 1243–1253.
- [Y1] S.-T. Yau, *Survey on partial differential equations in differential geometry*, Seminar on Differential Geometry, Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, NJ, 1982, 669–706.
- [Y2] S.-T. Yau, *Open problems in geometry*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. **54**, Part 1, Amer. Math. Soc., Providence, RI, 1993, 1–28.
- [Ze] S. Zelditch, Measure of nodal sets of analytic Steklov eigenfunctions, *Math. Res. Lett.* **22** (2015), no. 6, 1821–1842.
- [Zh1] J. Zhu, Doubling property and vanishing order of Steklov eigenfunctions, *Comm. Partial Differential Equations* **40** (2015), no. 8, 1498–1520.
- [Zh2] J. Zhu, Interior nodal sets of Steklov eigenfunctions on surfaces, *Analysis and PDE* **9** (2016), no. 4, 859–880.
- [Zh3] J. Zhu, Geometry and interior nodal sets of Steklov eigenfunctions, arXiv:1510.07300.
- [Zw] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics Series, Amer. Math. Soc., Providence, RI, 2012.

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL CP 6128 SUCC CENTRE-VILLE, MONTRÉAL QC H3C 3J7, CANADA.

E-mail address: `iossif@dms.umontreal.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 2074 EAST HALL, 530 CHURCH STREET, ANN ARBOR MI 48109-1043, U.S.A.

Current address: Department of Mathematical Sciences, DePaul University, 2320 N. Kenmore Ave., Chicago IL 60614, U.S.A.

E-mail address: `dsher@depaul.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, 805 SHERBROOKE STR. WEST, MONTRÉAL QC H3A 2K6, CANADA.

E-mail address: `jtoth@math.mcgill.ca`